Math 53, Second Midterm

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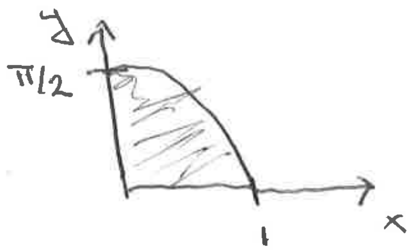
Instructions: *Please show your work: unjustified answers will not receive credit. Use back of page if needed. (No justification is required in the True/False section, however.) Your signature above certifies that the work here is your own.*

1. Evaluate $\int_0^1 \int_0^{\cos^{-1}(x)} e^{\sin y} dy dx$. [Recall that for $x \in [0,1]$ and $y \in [0, \frac{\pi}{2}]$, $\cos^{-1}(x) = y$ iff $\cos(y) = x$.]

$\int_0^1 \int_0^{\cos^{-1} x} e^{\sin y} \, dy \, dx = \iint_D e^{\sin y} \, dA$, where D is

the region:



The upper boundary curve is $y = \cos^{-1} x$, i.e. $\cos y = x$.

So

$$\iint_D e^{\sin y} \, dA = \int_0^{\pi/2} \int_0^{\cos y} e^{\sin y} \, dx \, dy$$

$$= \int_0^{\pi/2} \left[x e^{\sin y} \right]_0^{\cos y} dy$$

$$= \int_0^{\pi/2} e^{\sin y} \cos y \, dy \quad \left(\begin{array}{l} u = \sin y \\ du = \cos y \, dy \end{array} \right.$$

$$= \left[e^{\sin y} \right]_0^{\pi/2}$$

$$= e - 1$$

2. Find the centroid of E, where E is bounded by the surfaces $x^2 + y^2 = 1$, $z = x^2 + y^2$, and the plane $z = 0$. [The centroid is the center of mass for a constant density.]

$$\text{volume}(E) = \iiint_E 1\, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_0^{r^2} 1\, r\, dz\, dr\, d\theta$$

(in cylindrical coords, with $r^2 = x^2 + y^2$)

$$= 2\pi \int_0^1 [zr]_0^{r^2} dr = 2\pi \int_0^1 r^3 dr$$

$$= \pi/2$$

$\bar{x} = \bar{y} = 0$ by symmetry.

$$M_{xy} = \iiint_E z\, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_0^{r^2} z r\, dz\, dr\, d\theta$$

$$= 2\pi \int_0^1 \left[\tfrac{1}{2} z^2 r\right]_0^{r^2} dr = \frac{2\pi}{2} \int r^5 dr$$

$$= \pi \left[\tfrac{1}{6} r^6\right]_0^1 = \pi/6$$

$$\bar{z} = \frac{\pi/6}{\pi/2} = \frac{1}{3}$$

Centroid $= (0, 0, 1/3)$

3. The ellipsoidal shell $E = \{(x, y, z) \mid 1 \le \frac{x^2}{4} + \frac{y^2}{9} + z^2 \le 4 \text{ and } z \ge 0\}$ is filled with matter, with density $d(x, y, z) = z$. Find the total mass.

$m = \iiint_E z\,dV$. Transform to uvw-space by $x = 2u$, $y = 3v$, $z = w$. E transforms to D: $1 \le \frac{(2u)^2}{4} + \frac{(3v)^2}{9} + w^2 \le 4$, $w \ge 0$

or

$1 \le u^2 + v^2 + w^2 \le 4$, $w \ge 0$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 6$$

so $m = \iiint_D 6\,du\,dv\,dw$.

Now D is a "spherical shell", so using spherical coords:

$$m = 6\int_0^{2\pi}\int_0^{\pi/2}\int_1^2 \rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 12\pi\left(\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi\right)\left(\int_1^2 \rho^3\,d\rho\right)$$

$$= 12\pi\left(\frac{1}{2}\right)\left(4 - \frac{1}{4}\right) = 6\pi\left(\frac{15}{4}\right)$$

$$\boxed{m = \frac{45\pi}{2}}$$

4. Let $\vec{F}(x,y) = \frac{1}{x^2+y^2}\langle -y, x\rangle$. $= \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle = \langle P, Q \rangle$

(a) Show that $\vec{F}$ is conservative on D, where $D = \{(x,y) \mid \frac{1}{2} < x \text{ or } \frac{1}{2} < y\}$. What properties does D have that are relevant here?

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-1(x^2+y^2) - (-y\cdot 2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

So $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ on D, and D is open and simply connected. So $\vec{F}$ is conservative on D.

(b) Show that $\vec{F}$ is *not* conservative on $\mathbb{R}^2 - \{(0,0)\}$.

Let C = unit circle, oriented circlockwise. So param. C by $\vec{r}(t) = \langle \cos t, \sin t\rangle$, $t \in [0, 2\pi]$.
Then $\vec{r}\,'(t) = \langle -\sin t, \cos t\rangle$. So

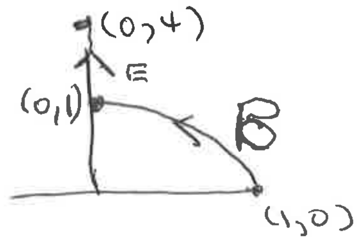
$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \frac{1}{\cos^2 t + \sin^2 t}\langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi.$$

Since C is closed, and $2\pi \neq 0$, $\vec{F}$ is not conservative on $\mathbb{R}^2 - \{(0,0)\}$.

(c) Find $\int_C \vec{F}\cdot d\vec{r}$, where C is any smooth curve in D from $(1,0)$ to $(0,4)$. [Hint: Use a convenient one, suggested by the solution of (b).]

Let $C = B + E$, where these are:



$$\int_C \vec{F}\cdot d\vec{r} = \int_B \vec{F}\cdot d\vec{r} + \int_E \vec{F}\cdot d\vec{r}$$

$$= \int_B \vec{F}\cdot d\vec{r} \quad (\text{since } \vec{F}\cdot d\vec{r} = 0 \text{ on } E)$$

But
$$\int_B \vec{F}\cdot d\vec{r} = \int_0^{\pi/2} \frac{1}{\cos^2 t + \sin^2 t}\langle -\sin t, \cos t\rangle \langle -\sin t, \cos t\rangle\, dt$$

$$= \int_0^{\pi/2} 1\, dt = \boxed{\pi/2}$$

5. **True or False.** (There is no penalty for guessing wrong.) In the questions below, a *nice* scalar or vector field is one which has partial derivatives of all orders which are continuous everywhere.

F 1. If C is an oriented smooth curve, and $-C$ is C with the opposite orientation, then $\int_C f\,ds = -\int_{-C} f\,ds$ for any nice scalar field f.

T 2. If f is continuous on $\mathbb{R}^2$, and R_a is the square with edge-length a centered at (x_0, y_0), then

$$f(x_0, y_0) = \lim_{a\to 0}\left(\frac{1}{a^2}\iint_{R_a} f\,dA\right).$$

F 3. Let S_1 and S_2 be type I regions in the uv-plane, and suppose they are mapped to regions R_1 and R_2 in the xy-plane, respectively, by the transformation $T(u,v) = (e^{u+v}, e^v)$. Then $\dfrac{\text{area}(R_1)}{\text{area}(S_1)} = \dfrac{\text{area}(R_2)}{\text{area}(S_2)}$.

T 4. The vector field $\vec{F}(\vec{r}) = \dfrac{\vec{r}}{|\vec{r}|^3}$ on $\mathbb{R}^3 - \{(0,0,0)\}$ is such that its line integrals are independent of path.

T 5. If h is a nice scalar field such that $\nabla h = \vec{0}$ everywhere in $\mathbb{R}^3$, then h is constant on $\mathbb{R}^3$.

T 6. Let C be a smooth curve in $\mathbb{R}^3$, and suppose g is a nice scalar field such that $g(x,y,z) = k$ for all points (x,y,z) on C. Then $\int_C \nabla g \cdot d\vec{r} = 0$.

Reasons

(2), (4), (6) : See sample T/F solutions.

(1) $\int_C f\,ds = \int_{-C} f\,ds$

(3) Jacobian is $\begin{vmatrix} e^{u+v} & e^{u+v} \\ 0 & e^v \end{vmatrix} = e^{(u+v)} \cdot e^v$. This is not constant, so area magnification depends on where you are.

(5) $0 = \int_C \nabla h \cdot d\vec{r} = h(B) - h(A)$, where