Physics 7B Midterm 2 Problem 1 Rubric

James Reed Watson

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a) (7 points) The electric field at point P is a superposition of the electric field generated from the four points, where the field from each point is found from Coulomb's Law.

$$\boldsymbol{E} = \frac{Q\hat{\boldsymbol{r}}}{4\pi\epsilon_0 r^2} \tag{1}$$

The distance for the bottom charges is $r = a\sqrt{1 + (\frac{1}{2})^2} = a\sqrt{\frac{5}{4}}$. Going counter clockwise from the lower left, the electric fields from each of the charges are

$$\boldsymbol{E_1} = \frac{-2Q}{4\pi\epsilon_0 a^2}(\frac{4}{5})^{\frac{3}{2}}(\frac{\hat{\boldsymbol{i}}}{2} + \hat{\boldsymbol{j}}) \tag{2}$$

$$\boldsymbol{E_2} = \frac{2Q}{4\pi\epsilon_0 a^2}(\frac{4}{5})^{\frac{3}{2}}(\frac{-\hat{\boldsymbol{i}}}{2} + \hat{\boldsymbol{j}}) \tag{3}$$

$$\boldsymbol{E_3} = \frac{4Q}{4\pi\epsilon_0 a^2}\hat{\boldsymbol{i}} \tag{4}$$

$$\boldsymbol{E_4} = \frac{4Q}{4\pi\epsilon_0 a^2}\hat{\boldsymbol{i}} \tag{5}$$

$$ \tag{6}$$

Adding these together, I get

$$\boldsymbol{E} = \frac{Q}{4\pi\epsilon_0}(2 * 4 - 2 * (\frac{4}{5})^{3/2})\hat{\boldsymbol{i}} = \frac{8Q}{4\pi\epsilon_0}(1 - \frac{2}{5^{3/2}})\hat{\boldsymbol{i}} \tag{7}$$

b) (7 points) To find the potential, I use the Coulomb potential, and the distances I already have to find that

$$V(P) = \frac{Q}{4\pi\epsilon_0 a}(2 - 2 + 2\sqrt{\frac{4}{5}} - 2\sqrt{\frac{4}{5}}) = 0 \tag{8}$$

c) (6 points) The electric dipole is found by

$$\boldsymbol{p} = \sum_i q_i \boldsymbol{r_i} \tag{9}$$

$$= Q(-a + 2a)\hat{\boldsymbol{i}} + Q(a - a)\hat{\boldsymbol{j}} \tag{10}$$

$$= Qa\boldsymbol{i} \tag{11}$$

I will award the points as follows, in a subtractive scheme:

1. -1 points for the incorrect answer on a) resulting only from algebraic mistake.

2. - 1 Forgot k on part a).
3. -1 Evaluated at the wrong point on a)
4. -3 points for the incorrect answer resulting from the incorrect application of Coulomb's Law (i.e. wrong power on r on a)
5. -2 points for the incorrect answer on a) resulting from not using vectors.
6. -1 point for incorrect answer on b) resulting only from a math error
7. -1 point for forgetting k on part b).
8. -1 point for evaluating at the wrong point on b)
9. -3 points for the incorrect answer resulting from the incorrect application of Coulomb's Law (i.e. wrong power on r on b)
10. -2 points on b) for using vectors when they shouldn't.
11. -3 points for incorrect answer on c) resulting only from a math error.
12. -3 points for incorrect expression/equation for dipole moment on c) that results in a wrong answer
13. -2 points for not being a vector on part c)
14. -6 for no attempt/answer on c)
15. -7 for no attempt/answer on b)
16. -7 for no attempt/answer on a)

Note that my total points add up to more than 20. Some of my points that I deduct are mutually exclusive, i.e. I won't take off points for a math error if they already made a mistake writing down Coulomb's law. Also, my rubrics for a) and b) are almost identical.

Fall 2016 - 7B Lectures 2 & 3 Midterm 2 - Q2 Solution

Question [20pts]

A block in the shape of a rectangular solid has a cross-sectional area A, a length L and a resistance R. The material of which the block is made has in total N conduction electrons. A potential difference of V_0 is maintained between it's ends.

(a) [7pts] Find the current in the block.
(b) [7pts] Assuming the current density is uniform, what is it's value?
(c) [6pts] Calculate the drift velocity of the conduction electrons and the electric field in the block.

Solution

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(a)

Ohm's Law for the current, I through a resistor R held at potential difference V is $V = IR$
So for this problem, $V = V_0$ and $R = R$ so the current is:

$$I = \frac{V_0}{R}$$

(b)

- The current density is defined as the current per cross sectional area.
- In this problem, the current I from part (a) passes through cross-sectional area A
- Assuming the current density is uniform, then it's magnitude is just the total current divided by A. It's direction will be the same direction as which the current flows. (As we shall find in part (c), this is the opposite direction to the direction in which electrons drift.)

$$\vec{j} = \frac{V_0}{AR}\hat{z}$$

Where $\hat{z}$ points in the direction that current flows, i.e perpendicular to the ends of the rectangular solid. The answer to this question, the value (or magnitude) of $\vec{j}$ is therefore:

$$j = \frac{V_0}{AR}$$

(c)

Drift Velocity:
- The current density, $\vec{j}$ is related to the drift velocity, $\vec{v}_d$ of electrons by the formula:

$$\vec{j} = -en\vec{v}_d$$

where $e = 1.602 \times 10^{-19}C$ is the charge on the electron and n is the *number density* of electrons in the material.

In this problem we are told the *total number of electrons*, N. We can find n by dividing N by the volume, which for a rectangular box is just AL.

$$n = \frac{N}{V} = \frac{N}{AL}$$

Thus, the drift velocity is found to be:

$$\begin{aligned}\vec{v}_d &= -\frac{\vec{j}}{en} = -\frac{V_0/AR}{eN/AL}\hat{z} \\ &= -\frac{V_0 L}{eNR}\hat{z}\end{aligned}$$

Note that this points in the opposite direction to $\vec{j}$.

Electric Field

The electric field may be found either by realising that for a resistor of length L and uniform cross-section, with a current flowing through it, the potential difference varies linearly with L and hence the electric field is just the change in potential over the length of the resistor:

$$E = \frac{V_0}{L}$$

and points in the direction of the current (or opposite to the direction in which electrons drift.)

Equivalently, albeit slightly more lengthily, we can use the microscopic form of Ohm's Law:

$$\vec{j} = \sigma \vec{E}$$

Where σ is the conductivity given by $\sigma = \frac{1}{\rho}$ where ρ is the resistivity.
In this problem, $\rho = \frac{RA}{L}$, so $\sigma = \frac{L}{RA}$ and so:

$$\vec{E} = \frac{\vec{j}}{\sigma} = \frac{V_0/AR}{L/RA}\hat{z}$$
$$= \frac{V_0}{L}\hat{z}$$

Rubric

Subtractive Grading

(a) [7pts]

[-0] Correct answer: $I = \frac{V_0}{R}$
[-5] Incorrect answer but recall's Ohm's law correctly or **very** nearly correctly.
[-7] No attempt / Incorrect starting point.

(b) [7pts]

[-0] Correct answer: Magnitude $j = \frac{V_0}{AR}$
[-4] Incorrect answer but correctly states current density is current divided by cross sectional area.
[-5] Incorrect but close definition of current density (e.g Current/Volume)
[-7] No attempt / Incorrect starting point.

(c) [6pts]

Drift Velocity [3pts]:
[-0] Correct answer for Drift Velocity: Magnitude $v_d = \frac{V_0 L}{eNR}$, directed opposite to the current.
[-1] Correct but omits that the drift velocity is in the opposite direction to the current.
[-1] Correct method with algebraic mistakes.
[-2] Correct starting point but no further progress,e.g states a correct definition of drift velocity.
[-3] No attempt / Incorrect starting point.
Electric Field [3pts]:
[-0] Correct answer for Electric Field: Magnitude $E = \frac{V_0}{L}$, directed along the current.
[-1] Correct answer for magnitude of electric field but direction not mentioned / wrong.
[-1] Correct method with algebra mistakes.
[-2] Correct starting point but no further progress, e.g writing down a correct relation between $\vec{E}$ and V or between $\vec{E}$ and $\vec{j}$.
[-3] No Attempt / Incorrect starting point.

Notes:
- In part (c), neglecting the direction of the vectors will only be penalised once.
- Full credit can be obtained in parts (b) and (c) if using a wrong answer from previous parts of the question correctly.

1 Solution to part a

$$V(x) = \frac{Q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{\left(x+\frac{L}{2}\right)^2+R^2}} - \frac{1}{\sqrt{\left(x-\frac{L}{2}\right)^2+R^2}}\right\} \tag{1}$$

(a) 2 points for point charge formula $V = Q/(4\pi\epsilon r)$

(b) 2 points (1 each) for the correct distances $\sqrt{(x+L/2)^2+R^2}$ and $\sqrt{(x-L/2)^2+R^2}$

(c) 4 points for the correct expression for V

2 Solution to part b

$$E = -\vec{\nabla}V \tag{2}$$

$$E_y = E_z = 0 \tag{3}$$

$$E_x = -\frac{\partial V}{\partial x} = \frac{Q}{4\pi\epsilon_0}\left\{\frac{x+\frac{L}{2}}{\sqrt{\left(x+\frac{L}{2}\right)^2+R^2}^3} - \frac{x-\frac{L}{2}}{\sqrt{\left(x-\frac{L}{2}\right)^2+R^2}^3}\right\} \tag{4}$$

$$\vec{E} = E_x\hat{x} \tag{5}$$

$$\tag{6}$$

(a) 1 point for the relation between E and V

(b) 1 point for Ey and Ez is zero from math

(c) 1 extra point for Ey and Ez is zero from symmetry

(d) up to 6 points for correct expression, -1 each for wrong: chain rule, derivative, sign, factor

3 Solution to part c

$$\vec{F} = q\vec{E} \tag{7}$$

$$\vec{F}(x=\frac{L}{2}) = q\vec{E}(x=\frac{L}{2}) = \frac{qQ}{4\pi\epsilon_0}\frac{L}{\sqrt{L^2+R^2}}\hat{x} \tag{8}$$

$$a = \frac{|\vec{F}|}{m} = \frac{qQ}{4m\pi\epsilon_0}\frac{L}{\sqrt{L^2+R^2}} \tag{9}$$

a is to the right if $q > 0$, to the left if $q < 0$

(a) 1 point for F=qE

(b) 2 points for correctly plugging $x = L/2$ into formula from part b (regardless of whether b was correct)

(c) 1 point for statement that direction depends on sign of q

Problem 4

1 A

We will use Gauss' law to find the electric field. Because of the axial symmetry (the system can be rotated about its axis and the charge distribution is unchanged) and translational symmetry along the axis I know that the electric field only depends on radius. Additionally, because I can reflect the system about a plane normal to the axis of the cylinder I know that my electric field will only have a radial component. In summary; $\vec{E} = E(r)\hat{r}$. Now I choose a Gauss surface of a cylinder with radius r, length l and will compute $\int \vec{E} \cdot d\vec{A}$. The integral over the caps of the surface is zero because the vectors are orthogonal. Because magnitude of E is constant at constant radius this integral is just $E(r)A = E(r)2\pi rl$ which by Gauss' law is $\frac{Q_{enc}}{\epsilon_0}$. For $r < R$ $Q_{enc} = \int \rho(r)dVol = b\int_0^r r(2\pi rl)dr = \frac{1}{2}2\pi blr^2$. For $r > R$ $Q_{enc} = b\int_0^R r(2\pi rl)dr = \frac{1}{2}2\pi blR^2$. After shuffling terms we get for $r < R$ $E(r) = \frac{br}{2\epsilon_0}$, for $r > R$ $E(r) = \frac{bR^2}{2\epsilon_0 r}$.

2 B

To find the speed of the particle after moving from a distance $2R$ to $3R$ we can find the change in kinetic energy which will by conservation of energy be the opposite of the difference in potential energy. $\Delta V = -\int \vec{E} \cdot d\vec{l} = -\frac{bR^2}{2\epsilon_0}\int_{2R}^{3R} r^{-1}dr = \frac{bR^2}{2\epsilon_0}\ln\frac{2}{3}$. $\Delta U = q\Delta V$ so $\Delta K = q\frac{bR^2}{2\epsilon_0}\log\frac{3}{2} = \frac{1}{2}mv^2$. Solving for v we find $v = \sqrt{\frac{q}{m}\frac{bR^2}{\epsilon_0}\ln\frac{3}{2}}$

For part (a): 2 points for a diagram including the cylinder and the gaussian surface showing (or stating) that the field is radial, 2 points for writing down Gauss law, 2 points each for evaluating the flux integral and charge enclosed and 2 points for using those results to get the electric field (split for the cases inside and outside the cylinder).

For part (b) If they integrate the electric field to get a change in voltage correctly is 4 points, recognizing that charge times change in voltage is change in potential is 1 point, writing down conservation of energy implies this is the change in kinetic energy is 3 points and then solving for the velocity from change in kinetic energy is 2 points.

1 Starting Rubric

(a) **(7)**

- **(3)**: Find the electric field in vacuum$\mathbf{E}_0(r)$ outside a charged conducting sphere with charge Q

$$\mathbf{E}_0(r) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}$$

OR give capacitance of a simple spherical capacitor with inner radius a and outer radius b

$$C_0 = 4\pi\epsilon_0\left(a^{-1} - b^{-1}\right)^{-1} = \frac{4\pi\epsilon_0 ab}{b-a}$$

- **(1)**: State that the field in the dielectric $\mathbf{E}_d(r) = \mathbf{E}_0(r)/K$ OR state the capacitance for a dielectric-filled spherical capacitor is $C_d = KC_0$.
- **(3)**: Get ΔV using

$$\Delta V = -\int \mathbf{E}(r)\cdot \mathrm{d}\boldsymbol{\ell}$$

and then $C_{\text{tot}} = Q/\|\Delta V\|$

OR use the series-capacitor argument $C_{\text{tot}} = \left(C_0^{-1} + C_d^{-1}\right)^{-1}$

(b) **(6)**

- **(3)**: Correct formula for constant V: $U = \frac{1}{2}CV^2$
- **(3)**: Plugging in C from part (a)

(c) **(7)**

- **(2)**: $\Delta Q = V(C_f - C_i)$
- **(2)**: Gave correct C_f
- **(3)**: Final answer is correct; partial if consistent with given C_f, C_i.

2 Solution

(a) The capacitance of a spherical capacitor is derived here. The vacuum electric field obtained by considering a Gaussian surface of radius r around the inner conducting sphere with charge Q on its surface. Gauss' law gives

$$\Phi_E = \oint \mathbf{E}\cdot \mathrm{d}\boldsymbol{A} = E\oint \mathrm{d}A = E4\pi r^2 \tag{1}$$

$$Q_{\text{encl}}/\epsilon_0 = Q/\epsilon_0 \tag{2}$$

$$\Rightarrow \mathbf{E}_0(r) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}} \tag{3}$$

The potential difference between the inner and outer shells with radii a and b, respectively, is then obtained by integrating along a path $\boldsymbol{\ell}$ from the outer radius to the inner radius:

$$\Delta V \equiv V(r=a) - V(r=b) = -\int \mathbf{E}(r)\cdot \mathrm{d}\boldsymbol{\ell} \tag{4}$$

$$= -\int \left(\frac{Q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}\right)\cdot \mathrm{d}\ell\hat{\boldsymbol{\ell}} \tag{5}$$

$$= -\int \left(\frac{Q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}\right)\cdot (-\mathrm{d}r)(-\hat{\boldsymbol{r}}) \tag{6}$$

$$= \frac{-Q}{4\pi\epsilon_0}\int_b^a \frac{\mathrm{d}r}{r^2} \tag{7}$$

$$\Delta V = \frac{Q}{4\pi\epsilon_0}\left(a^{-1} - b^{-1}\right) \tag{8}$$

Thus

$$C_0 = Q/\|\Delta V\| = C_0 = 4\pi\epsilon_0 \left(a^{-1} - b^{-1}\right)^{-1} = \frac{4\pi\epsilon_0 ab}{b-a} \tag{9}$$

Now, in a material,

$$C_d = KC_0$$

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The spherical capacitor in this problem is a series combination of C_d and C_0 with the corresponding inner and outer radii:

$$\begin{aligned} C_{\text{tot}} &= \left(C_d^{-1} + C_0^{-1}\right)^{-1} && (10)\\ &= 4\pi\epsilon_0 \left(\frac{1}{K}\left(a^{-1} - b^{-1}\right) + \left(b^{-1} - c^{-1}\right)\right)^{-1} && (11)\\ &= \frac{4\pi\epsilon_0 Kabc}{bc + a\left(c\left(K-1\right) - Kb\right)} && (12)\\ & && (13) \end{aligned}$$

or other algebraic equivalents.

(b) $U = \frac{1}{2}C_{\text{tot}}V^2$ is the appropriate formula, as V is held constant. Therefore,

$$U = \frac{V^2}{2} 4\pi\epsilon_0 \left(\frac{1}{K}\left(a^{-1} - b^{-1}\right) + \left(b^{-1} - c^{-1}\right)\right)^{-1} \tag{14}$$

(c)

$$\Delta Q = V\left(C_f - C_i\right)$$

where $C_i = C_{\text{tot}}$ was found in part (a) and $C_f = 4\pi\epsilon_0 K\left(a^{-1} - c^{-1}\right)^{-1}$

Thus,

$$\begin{aligned} \Delta Q &= V 4\pi\epsilon_0 \left(K\left(a^{-1} - c^{-1}\right)^{-1} - \left(\frac{1}{K}\left(a^{-1} - b^{-1}\right) + \left(b^{-1} - c^{-1}\right)\right)^{-1}\right) && (15)\\ &= V 4\pi\epsilon_0 \frac{a^2\left(b-c\right)c\left(K-1\right)K}{\left(a-c\right)\left(bc + a\left(c\left(K-1\right) - bK\right)\right)} && (16) \end{aligned}$$