Physics 137A, Spring 2004 , Section 1 (Hardtke), Midterm II Solutions

Problem 1 Part A Let,

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

The Hermitian conjugate is:

$$U^\dagger = \begin{pmatrix} U_{11}^* & U_{21}^* \\ U_{12}^* & U_{22}^* \end{pmatrix}$$

The unitarity condition, $U^\dagger U = \mathbf{1}$, gives:

$$\begin{pmatrix} U_{11}^* & U_{21}^* \\ U_{12}^* & U_{22}^* \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

These yield four equations that are conditions on the matrix elements:

$$U_{11}^* U_{11} + U_{21}^* U_{21} = 1$$

$$U_{11}^* U_{12} + U_{21}^* U_{22} = 0$$

$$U_{12}^* U_{11} + U_{22}^* U_{21} = 0$$

$$U_{12}^* U_{12} + U_{22}^* U_{22} = 1$$

Part B The equation $U_{11}^* U_{11} + U_{21}^* U_{21} = 1$ is equivalent to $R + T = 1$ for scattering from the left, and the equation $U_{12}^* U_{12} + U_{22}^* U_{22} = 1$ is equivalent to $R + T = 1$ for scattering from the right.

Problem 2 We are given $\hat{A}|\phi\rangle = a|\phi\rangle$ and

$$[\hat{A}, \hat{B}] = \hat{B} + 2\hat{B}\hat{A}^2$$

We can apply the commutation relation to the state $|\phi\rangle$:

$$[\hat{A}, \hat{B}]|\phi\rangle = (\hat{B} + 2\hat{B}\hat{A}^2)|\phi\rangle$$

Expanding the commutator, we have:

$$(\hat{A}\hat{B} - \hat{B}\hat{A})|\phi\rangle = (\hat{B} + 2\hat{B}\hat{A}^2)|\phi\rangle$$

We can move one term to the right side:

$$\hat{A}\hat{B}|\phi\rangle = (\hat{B} + 2\hat{B}\hat{A}^2 + \hat{B}\hat{A})|\phi\rangle$$

Using $\hat{A}|\phi\rangle = a|\phi\rangle$, we have,

$$\hat{A}(\hat{B}|\phi\rangle) = (1 + 2a^2 + a)(\hat{B}|\phi\rangle)$$

Thus $\hat{B}|\phi\rangle$ is and eigenvector of $\hat{A}$ with eigenvalue $(1 + 2a^2 + a)$.

Problem 3

$$\begin{aligned}
\frac{d\langle \hat{A}\rangle}{dt} &= \frac{d}{dt}\langle \Psi|\hat{A}\Psi\rangle \\
&= \langle \frac{\partial \Psi}{\partial t}|\hat{A}\Psi\rangle + \langle \Psi|\frac{\partial \hat{A}}{\partial t}\Psi\rangle + \langle \Psi|\hat{A}\frac{\partial \Psi}{\partial t}\rangle \\
&= \langle \frac{\partial \Psi}{\partial t}|\hat{A}\Psi\rangle + \langle \Psi|\hat{A}\frac{\partial \Psi}{\partial t}\rangle + \langle \frac{\partial \hat{A}}{\partial t}\rangle
\end{aligned}$$

Now we can use the Schr̈dinger equation,

$$\imath\hbar\frac{\partial \Psi}{\partial t} = \hat{H}\Psi,$$

and its complex conjugate (noting that $\hat{H}$ is a Hermitian operator),

$$-\imath\hbar\frac{\partial \Psi^*}{\partial t} = \hat{H}\Psi^*,$$

Thus,

$$\frac{\partial \Psi}{\partial t} = \frac{1}{\imath\hbar}\hat{H}\Psi$$

We now have,

$$\begin{aligned}
\frac{d\langle \hat{A}\rangle}{dt} &= \frac{-1}{\imath\hbar}\langle \hat{H}\Psi|\hat{A}\Psi\rangle + \frac{1}{\imath\hbar}\langle \Psi|\hat{A}\hat{H}\Psi\rangle + \langle \frac{\partial \hat{A}}{\partial t}\rangle \\
&= \frac{-1}{\imath\hbar}\langle \Psi|\hat{H}\hat{A}\Psi\rangle + \frac{1}{\imath\hbar}\langle \Psi|\hat{A}\hat{H}\Psi\rangle + \langle \frac{\partial \hat{A}}{\partial t}\rangle \\
&= \frac{1}{\imath\hbar}\langle \Psi|(\hat{A}\hat{H} - \hat{H}\hat{A})\Psi\rangle + \langle \frac{\partial \hat{A}}{\partial t}\rangle \\
&= \frac{1}{\imath\hbar}\langle [\hat{A}, \hat{H}]\rangle + \langle \frac{\partial \hat{A}}{\partial t}\rangle
\end{aligned}$$

In the second step, we have used the fact that $\hat{H}$ is Hermitian and can be moved to the second vector in the inner product. Multiplying by $\imath\hbar$ yields,

$$\imath\hbar\frac{d\langle \hat{A}\rangle}{dt} = \langle [\hat{A}, \hat{H}]\rangle + \imath\hbar\langle \frac{\partial \hat{A}}{\partial t}\rangle$$

Problem 4 Part A The stationary states $|\psi_n\rangle$ are solutions to the Schrödinger equation,

$$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle,$$

with $E_n = (n + \frac{1}{2})\hbar\omega$. The probability of measuring a given eigenvalue E_n is,

$$|c_n|^2 = |\langle \psi_n|\Psi\rangle|^2.$$

For,

$$|\Psi(x)\rangle = \frac{1}{\sqrt{5}}[2|\psi_2(x)\rangle + |\psi_3(x)\rangle],$$

we have $c_2 = 2/\sqrt{5}$ and $c_3 = 1/\sqrt{5}$ and all other c_n equal to zero. Thus we measure energy $E_2 = (2 + \frac{1}{2})\hbar\omega$ with probability $|c_2|^2 = \frac{4}{5}$, and energy $E_3 = (3 + \frac{1}{2})\hbar\omega$ with probability$|c_3|^2 = \frac{1}{5}$.

Part B We know that,

$$\hat{a}_+|\psi_n\rangle = c_n|\psi_{n+1}\rangle.$$

We thus have,

$$\begin{aligned}\langle\hat{a}_+\psi_n|\hat{a}_+\psi_n\rangle &= |c_n|^2\langle\psi_{n+1}|\psi_{n+1}\rangle \\ &= |c_n|^2\end{aligned}$$

In the last step, we use the fact that the $|\psi_n\rangle$ are orthonormal. We can also write the inner product as [using the relation $(\hat{a}_-)^\dagger = \hat{a}_+$],

$$\langle\hat{a}_+\psi_n|\hat{a}_+\psi_n\rangle = \langle\psi_n|\hat{a}_-\hat{a}_+\psi_n\rangle$$

We now note that the Hamiltonian can be written as,

$$\hat{H} = \hat{a}_-\hat{a}_+ - \frac{1}{2}\hbar\omega,$$

and thus,

$$\hat{a}_-\hat{a}_+ = \hat{H} + \frac{1}{2}\hbar\omega.$$

Making this substitution, we have:

$$\begin{aligned}\langle\hat{a}_+\psi_n|\hat{a}_+\psi_n\rangle &= \langle\psi_n|(\hat{H} + \frac{1}{2}\hbar\omega)\psi_n\rangle \\ &= \langle\psi_n|((n+\frac{1}{2})\hbar\omega + \frac{1}{2}\hbar\omega)\psi_n\rangle \\ &= (n+1)\hbar\omega\langle\psi_n|\psi_n\rangle.\end{aligned}$$

We now have $|c_n|^2 = (n+1)\hbar\omega$, or $c_n = \sqrt{(n+1)\hbar\omega}$.

Part C Using the result from Part B, we have,

$$\begin{aligned}|\Phi\rangle &= \hat{a}_+|\Psi(x)\rangle \\ &= \hat{a}_+(\frac{1}{\sqrt{5}}[2|\psi_2(x)\rangle + |\psi_3(x)\rangle]) \\ &= A(\frac{2}{\sqrt{5}}(\sqrt{3\hbar\omega})|\psi_3(x)\rangle + \frac{1}{\sqrt{5}}(\sqrt{4\hbar\omega})|\psi_4(x)\rangle)\end{aligned}$$

We introduce the complex number A in the last step since the vector is no longer normalized and we need to determine A. We get A from,

$$\begin{aligned}1 &= \langle\Phi(x)|\Phi(x)\rangle \\ &= |A|^2(\frac{2}{\sqrt{5}}(\sqrt{3\hbar\omega})\langle\psi_3(x)| + \frac{1}{\sqrt{5}}(\sqrt{4\hbar\omega})\langle\psi_4(x)|)(\frac{2}{\sqrt{5}}(\sqrt{3\hbar\omega})|\psi_3(x)\rangle + \frac{1}{\sqrt{5}}(\sqrt{4\hbar\omega})|\psi_4(x)\rangle) \\ &= |A|^2(\frac{4}{5}(3\hbar\omega) + \frac{1}{5}(4\hbar\omega)) \\ &= |A|^2\frac{16\hbar\omega}{5}\end{aligned}$$

This gives $A = \frac{\sqrt{5}}{4\sqrt{\hbar\omega}}$. Substituting into the equation for $|\Phi\rangle$ we get,

$$|\Phi\rangle = \frac{\sqrt{3}}{2}|\psi_3(x)\rangle + \frac{1}{2}|\psi_4(x)\rangle$$

Using the procedure from Part A, we measure energy $E = (3+\frac{1}{2})\hbar\omega$ with probability $|c_3|^2 = 3/4$ and measure energy $E = (4+\frac{1}{2})\hbar\omega$ with probability $|c_4|^2 = 1/4$.