Physics 7B

Final Exam: Monday December 14th, 2015

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Total points: 100 (7 problems)

Show all your work and take particular care to explain what you are doing. Partial credit can be given. Please use the symbols described in the problems or define any new symbol that you introduce. Label any drawings that you make. ***Good luck!***

Problem 1 (15 pts)

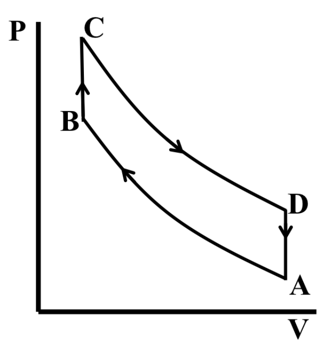
The Otto cycle (which most gas engines use) comprises four segments: (1) adiabatic compression from $A \to B$, (2) isovolumetric heat addition from $B \to C$, (3) adiabatic expansion from $C \to D$, and (4) isovolumetric heat release from $D \to A$.

On the other hand, the Diesel cycle (which Diesel engines use), comprises the following four segments: (1) adiabatic compression from $E \to F$, (2) isobaric heat addition from $F \to G$, (3) adiabatic expansion from $G \to H$, and (4) isovolumetric heat release from $H \to E$.

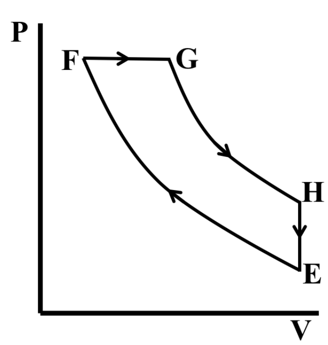
(a) (5 pts) Derive their efficiencies in terms of the the ratio of specific heats $\gamma = C_P/C_V$ and the respective temperatures (T_A, T_B, T_C, T_D) and (T_E, T_F, T_G, T_H).

(b) (5 pts) Derive the efficiency η_O of the Otto cycle in terms of the ratios V_A/V_B, V_A/V_C, and γ.

(c) (5 pts) Show that the efficiency of the Diesel cycle is $\eta_D = 1 - \frac{1}{\gamma}\frac{(V_E/V_G)^{-\gamma} - (V_E/V_F)^{-\gamma}}{(V_E/V_G)^{-1} - (V_E/V_F)^{-1}}$



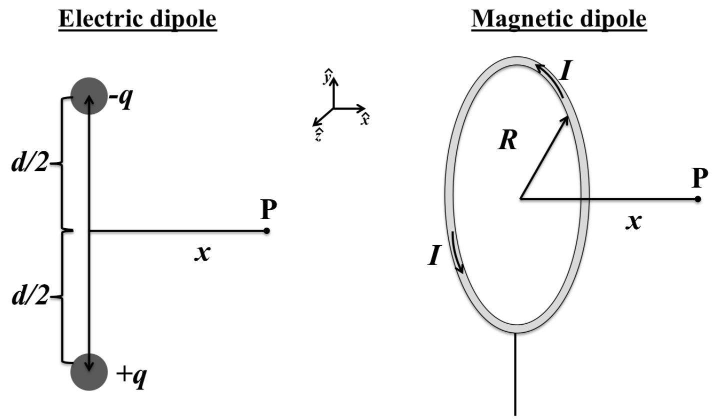
Otto cycle



Diesel cycle

Problem 2 (15 pts)

In this problem, we will derive and compare the strengths of electric and magnetic fields produced by dipoles, as shown below.



(a) (7.5 pts) Calculate the magnitude and direction of the electric field produced by an electric dipole (two charges $+q$ and $-q$ separated by a distance d) at a point P located at a distance x perpendicular to the dipole moment, taking the limit $x >> d$. Express the electric field as function of the magnitude of the electric dipole moment $|p|$.

(b) (7.5 pts) Calculate the magnitude and direction of the magnetic field produced by a magnetic dipole (a current I flowing counterclockwise through a circular loop of radius R) at a point P located at a distance x along the loop's axis, taking the limit $x >> R$. Express the magnetic field as function of the magnitude of the dipole moment $|\mu|$.

Problem 3 (15 pts)

The hydrogen atom can be modeled by considering a spherical negative charge distribution $\rho(r)$ around the proton of charge $q > 0$. At distance r from the center O of the atom, the electric potential is given by the following expression:

$$V(r) = \frac{q}{4\pi\epsilon_0 r} e^{-\frac{r}{a}} \tag{1}$$

where a is a positive constant.

(a) (5 pts) Determine the direction and magnitude of the electric field $\vec{E}(r)$ created by this charge distribution at a distance r from the origin.

(b) (5 pts) Calculate the flux $\Phi_E(r)$ of the electric field through a sphere of center O and radius r.

(c) (5 pts) Calculate the electric charge Q_i enclosed in the sphere of center O and radius r. What is the limit of Q_i when $r \to \infty$? Interpret your result.

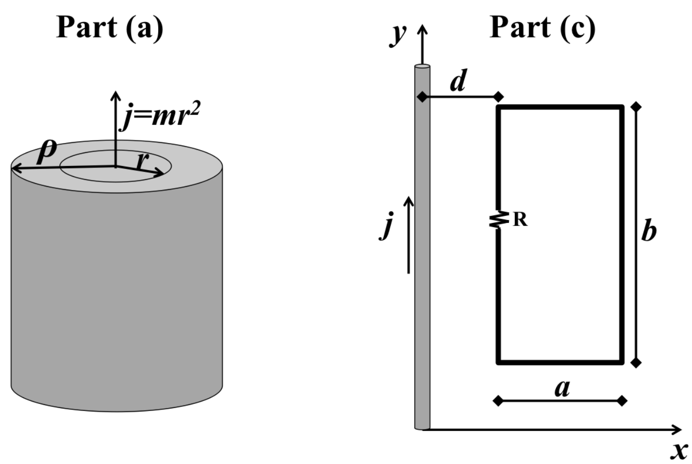
Problem 4 (15 pts)

A long straight wire of radius ρ carries a current along its axis with a non-uniform current density $j(r) = mr^2$ (m=constant), r being the radial distance measured from the symmetry axis of the wire, as shown below.

(a) (5 pts) Calculate the magnitude of the magnetic field produced inside and outside the wire.

(b) (2 pts) Draw some field lines to show qualitatively how the magnitude and direction of the magnetic field vary. Specify the direction of the current in your drawing.

(c) (8 pts) A rectangular loop of sides a and b and resistance R is placed at distance d ($d > \rho$) from the center of the current-carrying wire, as shown below. What is the induced current in the loop if it is (i) translated along the y-axis at constant speed v? (ii) translated along the x-axis at speed v?



Problem 5 (10 pts)

In this problem, we will discuss the energy stored in an inductor.

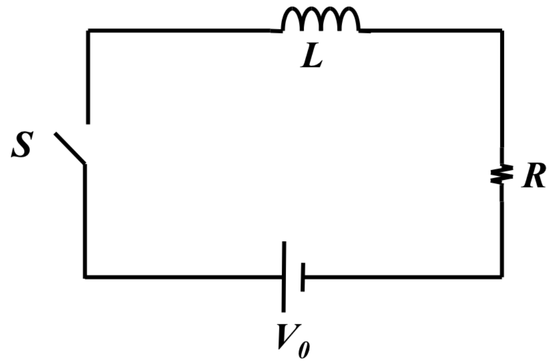
(a) (4 pts) Consider a tightly wrapped and long solenoid containing N turns of wire in its length l, whose cross section area is A. What is the self-inductance L of this solenoid in terms of the given variables?

(b) (3 pts) Knowing that when an inductor of inductance L carries a time varying current $I(t)$, the power supplied to the inductor is $P = I\mathcal{E}$, derive an expression for the energy U stored in an inductor as function of L and I. Assume that $U = 0$ when the current is still not flowing.

(c) (3 pts) Derive the energy per unit volume stored in the solenoid of part (a) in terms of the magnetic field strength B?

Problem 6 (15 pts)

Consider the LR circuit shown in the figure below.



(a) (5 pts) Derive the expression for the current as function of the battery potential V_0, resistance R, time constant τ, and time t after closing the switch at $t = 0$. Show and explain all your steps.

At $t = 0$, a $10V$ battery is connected in series with a $0.1H$ inductor and a total of 10Ω resistance.

(b) (2 pts) What is the current at $t = 0$, i.e. when the switch is closed?

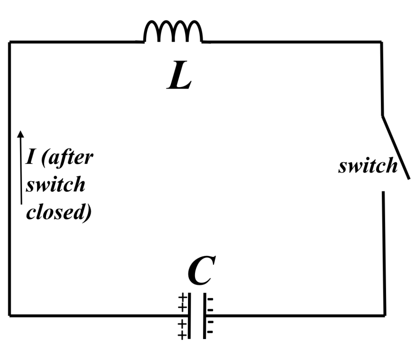
(c) (2 pts) What is the time constant τ?

(d) (2 pts) What is the maximum current?

(e) (2 pts) At the maximum current, at what rate is energy being delivered by the battery?

(f) (2 pts) At the maximum current, at what rate is energy being stored in the inductor's magnetic field?

Problem 7 (15 pts) Consider the LC circuit shown in the figure below, in which the capacitor has charge Q_0 at $t \leq 0$.



(a) (3 pts) At $t = 0$, the circuit is closed. Derive the differential equation, containing L, C, I and the charge Q in the capacitor, that describes what happens throughout the circuit. What principle is used to derive this?

(b) (2 pts) Express the previous differential equation in a form such that it resembles that of a harmonic oscillator. What physical quantity is oscillating in time?

(c) (2 pts) What is the oscillation frequency ω in terms of L, C?

(d) (2 pts) Derive the expression for the current as function of time.

(e) (2 pts) Plot the charge $Q(t)$ and current $I(t)$ as functions of time. What is the phase relationship between the two?

(f) (2 pts) Calculate the energy stored in the electric field of the capacitor $U_E(t)$ as well as the energy stored in the magnetic field of the inductor $U_B(t)$, both as function of time.

(g) (2 pts) What is the total energy of the system? Explain the time dependence.

$$PV^\gamma = \text{const. (For an adiabatic process)}$$

$$W = -\frac{d}{2}(P_f V_f - P_0 V_0)$$
(For an adiabatic process)

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$$

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{l} \times \hat{r}}{r^2}$$

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$\vec{E} = -\vec{\nabla} V$$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$
$$d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$
(Cartesian Coordinates)

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{z}$$
$$d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + dz\hat{z}$$
(Cylindrical Coordinates)

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \phi}\hat{\phi}$$
$$d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin(\theta)d\phi\hat{\phi}$$
(Spherical Coordinates)

$$y(t) = \frac{B}{A}(1 - e^{-At}) + y(0)e^{-At}$$
solves $\frac{dy}{dt} = -Ay + B$

$$y(t) = y_{max}\cos(\sqrt{A}t + \delta)$$
solves $\frac{d^2y}{dt^2} = -Ay$

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$\int_0^\infty x^{2n} e^{-ax^2} dx = \frac{(2n)!}{n! 2^{2n+1}}\sqrt{\frac{\pi}{a^{2n+1}}}$$

$$\int_0^\infty x^{2n+1} e^{-ax^2} dx = \frac{n!}{2a^{n+1}}$$

$$\int_0^\pi \sin^3(x) dx = \frac{4}{3}$$

$$\int (1+x^2)^{-1/2} dx = \ln(x + \sqrt{1+x^2})$$

$$\int (1+x^2)^{-1} dx = \arctan(x)$$

$$\int (1+x^2)^{-3/2} dx = \frac{x}{\sqrt{1+x^2}}$$

$$\int \frac{x}{1+x^2} dx = \frac{1}{2}\ln(1+x^2)$$

$$\int \frac{1}{\cos(x)} dx = \ln\left(\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right|\right)$$

$$\int \frac{1}{\sin(x)} dx = \ln\left(\left|\tan\left(\frac{x}{2}\right)\right|\right)$$

$$\int \sin(x) dx = -\cos(x)$$

$$\int \cos(x) dx = \sin(x)$$

$$\int \frac{dx}{x} = \ln(x)$$

$$\sin(x) \approx x$$

$$\cos(x) \approx 1 - \frac{x^2}{2}$$

$$e^x \approx 1 + x + \frac{x^2}{2}$$

$$(1+x)^\alpha \approx 1 + \alpha x + \frac{(\alpha - 1)\alpha}{2}x^2$$

$$\ln(1+x) \approx x - \frac{x^2}{2}$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = 2\cos^2(x) - 1$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$1 + \cot^2(x) = \csc^2(x)$$

$$1 + \tan^2(x) = \sec^2(x)$$