**EE 16B Midterm 2, October 25, 2016**

Name: SOLUTIONS

SID #: ____________

Discussion Section and TA: ____________

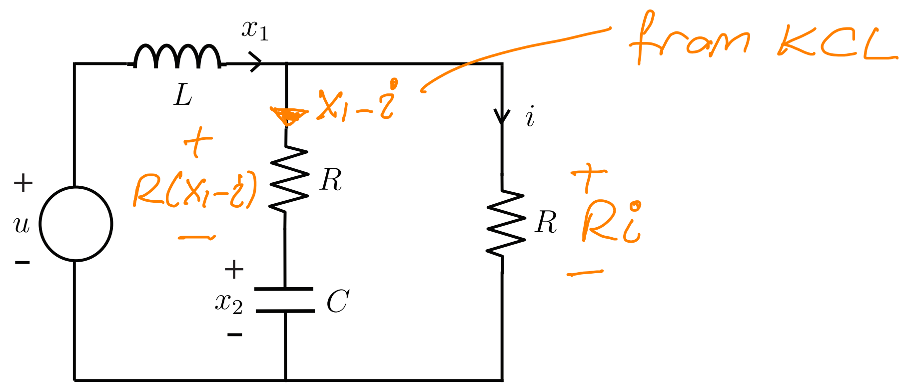
Lab Section and TA: ____________

**Important Instructions:**

- **Show your work.** An answer without explanation is not acceptable and does not guarantee any credit.
- **Only the front pages will be scanned and graded.** If you need more space, please ask for extra paper instead of using the back pages.
- **Do not remove pages**, as this disrupts the scanning. Instead, cross the parts that you don't want us to grade.

| Problem | Points | Score |
|---|---|---|
| 1 | 15 | |
| 2 | 10 | |
| 3 | 15 | |
| 4 | 25 | |
| 5 | 20 | |
| 6 | 15 | |
| Total | 100 | |

1. (15 points) Consider the circuit below that consists of two *identical* resistors, an inductor, a capacitor, and a voltage source whose voltage at time $t$ is $u(t)$.




a) (5 points) Write an expression for the current $i$ indicated on the circuit diagram in terms of $x_1$ (the inductor current) and $x_2$ (the capacitor voltage). Your answer should **not** involve any derivatives.

From KVL: $R(x_1 - i) + x_2 = Ri$

$$2Ri = Rx_1 + x_2$$

$$i = \frac{Rx_1 + x_2}{2R}$$

$$\boxed{i = \frac{1}{2}x_1 + \frac{1}{2R}x_2}$$

b) (10 points) Write a state space model using the states $x_1(t)$, $x_2(t)$, and input $u(t)$. Your final answer should specify the $A$ and $B$ matrices, with entries that depend on $R$, $L$, $C$.

KVL for outer loop gives:

$$\underbrace{L\frac{dx_1}{dt}}_{\text{inductor voltage}} + Ri = u$$

$$Ri = \frac{R}{2}x_1 + \frac{1}{2}x_2 \text{ from part (a)}$$

$$L\frac{dx_1}{dt} = -\frac{R}{2}x_1 - \frac{1}{2}x_2 + u$$

$$\frac{dx_1}{dt} = \frac{-R}{2L}x_1 - \frac{1}{2L}x_2 + \frac{1}{L}u$$

For the other ~~state~~ state we have: again, from part (a)

$$\underbrace{C\frac{dx_2}{dt}}_{\text{capacitor current}} = x_i - i = x_1 - \left(\frac{1}{2}x_1 + \frac{1}{2R}x_2\right)$$

$$= \frac{1}{2}x_1 - \frac{1}{2R}x_2$$

$$\frac{dx_2}{dt} = \frac{1}{2C}x_1 - \frac{1}{2RC}x_2$$

Combine in matrix form:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{R}{2L} & \frac{-1}{2L} \\ \frac{1}{2C} & \frac{-1}{2RC} \end{bmatrix}}_{=A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}}_{=B} u$$

Additional workspace for Problem 1b.

2. (10 points) Consider the scalar discrete-time system

$$x(t+1) = f(x(t))$$

where

$$f(x) = 2x - 2x^2.$$

a) (2 points) What is the solution $x(t)$ for $t > 0$ if $x(0) = 0.5$?

$f(0.5) = 2(0.5) - 2(0.5)^2 = 0.5$

Therefore $x(1) = f(x(0)) = f(0.5) = 0.5$

$x(2) = f(x(1)) = f(0.5) = 0.5$

$\vdots$

$\boxed{x(t) = 0.5 \text{ for all } t \geq 0.}$

b) (3 points) Find **all** equilibrium points of the system.

Equilibrium points of a discrete-time system are the solutions of $f(x) = x$:

$2x - 2x^2 = x$

$\Rightarrow 2x^2 - x = 0$

$x(2x-1) = 0 \longrightarrow$ $\boxed{x = 0 \text{ and } x = 0.5}$

c) (5 points) Linearize the system around each equilibrium and determine stability for the resulting linear models.

$$f(x) = 2x - 2x^2$$

$$\frac{\partial f}{\partial x} = 2 - 4x$$

For equilibrium @ $x=0$:

$$\boxed{A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = 2}$$

UNSTABLE because 2 is outside unit circle

For equilibrium @ $x=0.5$:

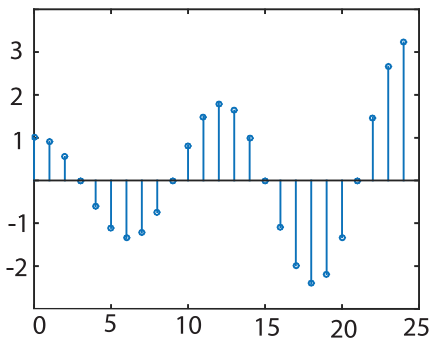
$$\boxed{A = \left.\frac{\partial f}{\partial x}\right|_{x=0.5} = 0}$$

STABLE because 0 is inside unit circle

3. (15 points) Each plot below shows $x_1(t)$ obtained from the solution of

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

where $A$ is one of the matrices below. Match each matrix to a plot and write the corresponding letter (a, b, c, or d) in the box next to each plot.



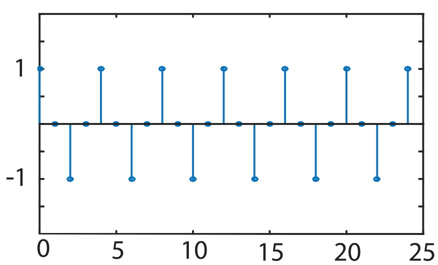
$\boxed{d}$

(a) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1 = 0$

$\lambda_{1,2} = \mp i$

$\lambda^t = (\mp i)^t$ oscillates with period $=4$



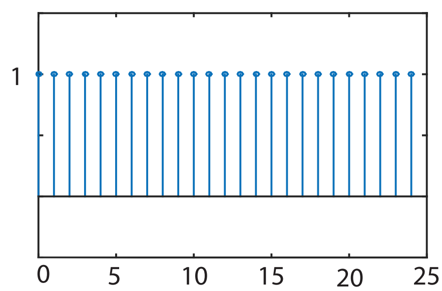
$\boxed{a}$

(b) $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda-2 \end{vmatrix} = \lambda^2 - 2\lambda + 1 = 0$

$\lambda_{1,2} = 1$

$\lambda^t = 1$ for all $t$



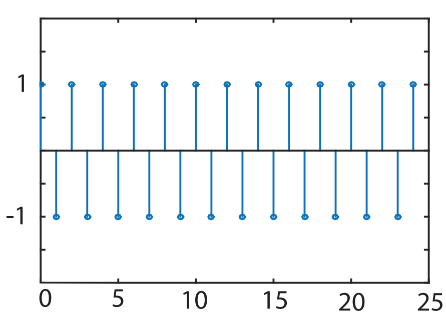
$\boxed{b}$

(c) $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda+2 \end{vmatrix} = \lambda^2 + 2\lambda + 1 = 0$

$\lambda_{1,2} = -1$

$\lambda^t = (-1)^t$ oscillates with period $=2$



$\boxed{c}$

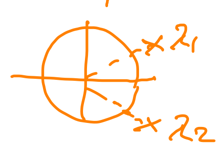
(d) $A = 1.05 \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$

Eigenvalues of $A$ are $1.05$ times the eigenvalues of

$\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \rightarrow \begin{vmatrix} \lambda - \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \lambda - \frac{\sqrt{3}}{2} \end{vmatrix}$

$= \left(\lambda - \frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 0$

$\lambda = \frac{\sqrt{3}}{2} \mp i\frac{1}{2}$




scale by 1.05

$\lambda_{1,2} = 1.05\left(\frac{\sqrt{3}}{2} \mp i\frac{1}{2}\right)$

Additional workspace for Problem 3.

4. (25 points) Consider the system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 1.5 & 1 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t).$$

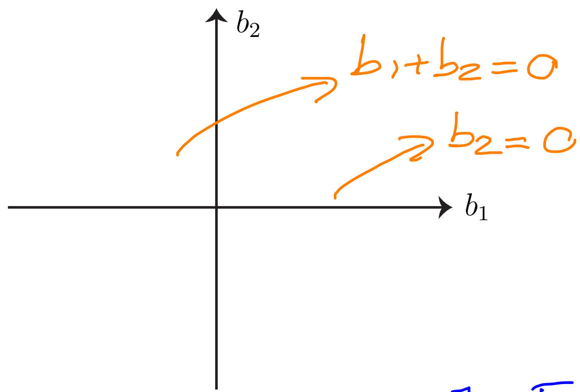
a) (5 points) Determine if the system is stable.

Evales of $A = \begin{bmatrix} 1.5 & 1 \\ 0 & 0.5 \end{bmatrix}$ are 1.5 and 0.5.

outside unit circle

$\boxed{\text{UNSTABLE}}$

b) (5 points) Determine the set of all $(b_1, b_2)$ values for which the system is **not** controllable and sketch this set of points in the $b_1$-$b_2$ plane below.




$$B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad AB = \begin{bmatrix} 1.5 & 1 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1.5\,b_1 + b_2 \\ 0.5\,b_2 \end{bmatrix}$$

$$[AB \mid B] = \begin{bmatrix} 1.5\,b_1 + b_2 & b_1 \\ 0.5\,b_2 & b_2 \end{bmatrix}$$

is rank deficient if

$$b_2(1.5\,b_1 + b_2) - 0.5\,b_1 b_2 = 0$$

$$\Rightarrow \quad b_1 b_2 + b_2^2 = 0$$

$$\Rightarrow \quad b_2(b_1 + b_2) = 0 \quad \Rightarrow \quad \boxed{\begin{matrix} b_2 = 0 \\ \text{or} \\ b_1 + b_2 = 0 \end{matrix}}$$

c) (6 points) Suppose $b_1 = 1$ and $b_2 = 0$. Design a state feedback controller such that the closed-loop system is stable.
*Note:* The problem doesn't prescribe specific eigenvalues, other than that they be in the stable region. Therefore controllability is not a necessary condition.

$$A + BK = \begin{bmatrix} 1.5 & 1 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 + k_1 & 1 + k_2 \\ 0 & 0.5 \end{bmatrix}$$

Evalues: $1.5 + k_1$ and $0.5$

Need to move this inside unit circle: choose $k_1$ such that

$$\boxed{|1.5 + k_1| < 1}$$

for example, $k_1 = -1$

We can't change this (uncontrollable) but it is already inside unit circle

d) (9 points) Suppose $b_1 = 0$ and $b_2 = 1$. Design a state feedback controller such that the closed-loop system eigenvalues are $\lambda_1 = \lambda_2 = 0.5$.

$$A+BK = \begin{bmatrix} 1.5 & 1 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 & 1 \\ k_1 & 0.5+k_2 \end{bmatrix}$$

Evalues from $\begin{vmatrix} \lambda-1.5 & -1 \\ -k_1 & \lambda-(0.5+k_2) \end{vmatrix} = 0$

$$\lambda^2 - (2+k_2)\lambda + 1.5(0.5+k_2) - k_1 = 0.$$

Match coefficients to

$$(\lambda-\lambda_1)(\lambda-\lambda_2) = (\lambda-0.5)^2 = \lambda^2 - \lambda + 0.25.$$

$$-(2+k_2) = -1 \quad \Rightarrow \quad \boxed{k_2 = -1}$$

$$1.5(0.5+k_2) - k_1 = 0.25$$

$$1.5(0.5-1) - k_1 = 0.25$$

$$-0.75 - k_1 = 0.25 \quad \Rightarrow \quad \boxed{k_1 = -1}$$

Additional workspace for Problem 4d.

5. (20 points) Consider the system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$
$$y(t) = x_1(t).$$

a) (4 points) Show that the system is observable.

$y = x_1$ means $y = \underbrace{[1 \ \ 0]}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ has full rank

$\Rightarrow$ OBSERVABLE

b) (6 points) Suppose we measure $y(t) = x_1(t)$ at $t = 0$ and $t = 1$, and find

$$y(0) = 1, \qquad y(1) = 0.$$

Determine the unmeasured state $x_2(t)$ at $t = 0$ and $t = 1$:

$\boxed{x_2(0) = 1}$ $\boxed{x_2(1) = 2}$

From state equation

$y(1) = x_1(1) = x_1(0) - x_2(0)$
$= y(0) - x_2(0) \Rightarrow x_2(0) = y(0) - y(1)$
$= 1$

Therefore initial condition is

$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Additional workspace for Problem 5b.

c) (10 points) Select values for $l_1$ and $l_2$ in the observer below such that $\hat{x}_1(t)$ and $\hat{x}_2(t)$ are guaranteed to converge to $x_1(t)$ and $x_2(t)$. (You are free to choose appropriate eigenvalues that guarantee convergence.)

$$\begin{bmatrix} \hat{x}_1(t+1) \\ \hat{x}_2(t+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{=A} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}}_{L} (\hat{x}_1(t) - y(t))$$

$$A + LC = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+l_1 & -1 \\ 1+l_2 & 1 \end{bmatrix}$$

Evalues from

$$\begin{vmatrix} \lambda-(1+l_1) & 1 \\ -(1+l_2) & \lambda-1 \end{vmatrix} = \lambda^2 - (2+l_1)\lambda + (2+l_1+l_2) = 0$$

Match coefficients to

$$(\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2$$

where $\lambda_1$ and $\lambda_2$ are your choice (they must satisfy $|\lambda_1| < 1$ and $|\lambda_2| < 1$ for convergence).

Then: $\quad 2 + l_1 = \lambda_1 + \lambda_2 \quad \Rightarrow \quad l_1 = \lambda_1 + \lambda_2 - 1$

$\quad 2 + l_1 + l_2 = \lambda_1\lambda_2 \quad \Rightarrow \quad l_2 = \lambda_1\lambda_2 - 2 - l_1$

$\quad = \lambda_1\lambda_2 - 1 - (\lambda_1+\lambda_2)$

Additional workspace for Problem 5c.

6. (15 points) Suppose we have two systems,

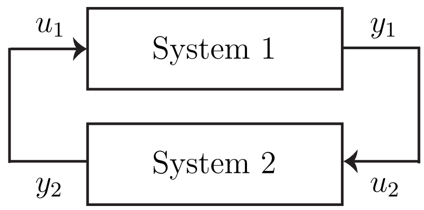
$$\text{System 1:} \quad \vec{x}_1(t+1) = A_1\vec{x}_1(t) + B_1u_1(t), \quad y_1(t) = C_1\vec{x}_1(t),$$

$$\text{System 2:} \quad \vec{x}_2(t+1) = A_2\vec{x}_2(t) + B_2u_2(t), \quad y_2(t) = C_2\vec{x}_2(t),$$

and connect the output of the first to the input of the second, and vice versa:

$$u_1(t) = y_2(t) \quad \text{and} \quad u_2(t) = y_1(t).$$

The dimensions of the states, inputs, and outputs above are arbitrary, except that the output of one system must have the same dimension as the input of the other. The resulting interconnection is shown in the block diagram below.




a) (5 points) Fill in the four blocks of the matrix below which describes the combined state model.

$$\begin{bmatrix} \vec{x}_1(t+1) \\ \vec{x}_2(t+1) \end{bmatrix} = \left[\begin{array}{c|c} A_1 & B_1C_2 \\ \hline B_2C_1 & A_2 \end{array}\right] \begin{bmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{bmatrix}.$$

$$\begin{aligned} \vec{x}_1(t+1) &= A_1\vec{x}_1(t) + B_1u_1(t) \\ &= A_1\vec{x}_1(t) + B_1y_2(t) \\ &= A_1\vec{x}_1(t) + B_1C_2\vec{x}_2(t) \end{aligned}$$

$$\begin{aligned} \vec{x}_2(t+1) &= A_2\vec{x}_2(t) + B_2u_2(t) \\ &= A_2\vec{x}_2(t) + B_2y_1(t) \\ &= A_2\vec{x}_2(t) + B_2C_1\vec{x}_1(t) \end{aligned}$$

b) (10 points) Show that stability of both System 1 and System 2 does **not** guarantee stability for the interconnection.

*Hint:* Construct an example where $A_1$ and $A_2$ each satisfy the discrete-time stability condition, but $B_1$, $B_2$, $C_1$, $C_2$ are such that the matrix you found in part (a) fails the stability condition.

To create a simple example we can take $A_1, A_2, B_1, B_2, C_1, C_2$ to be scalars.

Let $|A_1|<1$, $|A_2|<1$ for stability of Systems 1 and 2. To make the interconnection unstable look for $B_1, B_2$ $C_1, C_2$ such that

$$\begin{bmatrix} A_1 & B_1C_2 \\ B_2C_1 & A_2 \end{bmatrix}$$

has eigenvalues outside unit circle.

For example, $A_1=A_2=0$, $B_1=B_2=2$, $C_1=C_2=1$ give

$$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

whose evalues are $\mp 2$, both outside the unit circle.

Additional workspace for Problem 6b.