Midterm 1 Solutions

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1 Problem 1

1. We need $c \geq 0$

2. Marginal rate of substitution is $\frac{dy}{dx} = \frac{u_x}{u_y} = \frac{cy^{1/2}}{x^{1/2}} = c\sqrt{\frac{y}{x}}$. Intuitively, this is an "exchange rate." Holding utility constant, how much of y do I need to get if I'm going to give up a unit of x.

3. $\max_{x,y} u(x,y) = c\sqrt{x} + \sqrt{y}$ s.t. $p_x x + y \leq B$. Even though these won't end up mattering, we also apply non-negativity constraints $x, y \geq 0$. Aiden's choice variables are x and y. The parameters are c, p_x, and B.

4. If the budget did not hold with equality at the optimum (x_1, y_1), then one could use the budget left $(b = B - (p_x x_1 + y_1) > 0)$ and buy with it some popcorn and/or pretzel. This would increase utility, since preferences are monotonic (if I have $x_1 \geq x_2$ and $y_1 \geq y_2$ then $c\sqrt{x_1} + \sqrt{y_1} \geq c\sqrt{x_2} + \sqrt{y_2}$). Therefore a solution with some budget left unused can't be an optimum.

5. $$\max_{x,y} \mathcal{L}(x,y) = c\sqrt{x} + \sqrt{y} - \lambda(p_x x + y - B)$$

   which yields first order conditions

   $$\begin{aligned} 0 &= -p_x x^* - y^* + B \\ 0 &= \frac{1}{2}\frac{c}{\sqrt{x^*}} - \lambda^* p_x \\ 0 &= \frac{1}{2}\frac{1}{\sqrt{y^*}} - \lambda^* \end{aligned}$$

6. Solving the second two first order conditions for x^*, I get

   $$\begin{aligned} \frac{1}{2p_x}\frac{c}{\sqrt{x^*}} &= \frac{1}{2}\frac{1}{\sqrt{y^*}} \\ p_x^2 x^* &= c^2 y^* \\ x^* &= \frac{c^2}{p_x^2} y^* \end{aligned}$$

and then plugging back into the first first order condition (the budget constraint), I get

$$
\begin{aligned}
0 &= \frac{c^2 y^*}{p_x} + y^* - B \\
y^* &= \frac{p_x}{c^2 + p_x} B
\end{aligned}
$$

and then plugging back into my formula for x^*:

$$
\begin{aligned}
x^* &= \frac{c^2 p_x}{c^2 p_x^2 + p_x^3} B \\
&= \frac{c^2}{c^2 p_x + p_x^2} B = \frac{c^2}{c^2 + p_x} \cdot \frac{B}{p_x}
\end{aligned}
$$

7. Here we ask if the Marshallian demand changes as the relative prices change, holding the budget constant. One can see $\frac{\partial y^*}{\partial p_x} = \frac{c^2}{(c^2+p_x)^2} B > 0$, and therefore popcorn is a gross substitute of pretzels. This does not necessarily mean that pretzels are a gross substitute of popcorn based on parameterization of the problem—one need only acknowledge this possibility to get full credit. If we were to allow p_y to be a parameter of the problem, we would find that indeed pretzels are a gross substitute of popcorn.

8. If you only have two goods, they are always net substitutes. This is easiest to see graphically, but one way to think about this is when you hold utility constant, you're only concerned about *relative prices*. Increasing the price of good 2 p_2 is the same as lowering the price of p_1, which will always increase the Hicksian demand of good 1 h_1^*. The reason why this doesn't work for multiple goods because when you raise the price p_2, the relative prices $\frac{p_2}{p_1}$ and $\frac{p_2}{p_3}$ go up, but the relative price $\frac{p_1}{p_3}$ stays the same.

9. The optimal quantity of popcorn x^* increases with B, and therefore popcorn is a normal good.

10. Aidan's budget share for popcorn is
$$\frac{p_x x^*}{B} = \frac{c^2}{c^2 + p_x}$$
Popcorn is not a luxury good because it stays constant with income. While Cobb-Douglas shares this particular property, the budget shares are just constant, not a function of price. Because of that, you can't represent these preferences using a Cobb-Douglas utility function.

11. Since popcorn is a normal good, it cannot be a Giffen good. For it to be a Giffen good, the income effect would have to be positive (positive enough to wash out the substitution effect), so the good would have to be an inferior good. You can use the Slutsky equation and observe that the substitution effect is negative (always) and the income effect is negative (part 8).

12. We could say that advertising may affect c. Differentiating our expression for x^* with respect to c, we get:
$$
\begin{aligned}
\frac{\partial x^*}{\partial c} &= \frac{2c\left(c^2 p_x + p_x^2\right) - c^2 \cdot 2c p_x}{\left(c^2 p_x + p_x^2\right)^2} B \\
&= \frac{2c p_x^2}{\left(c^2 p_x + p_x^2\right)^2} > 0
\end{aligned}
$$

Consumers whose utility have a higher c spend more on good x, all else equal (that is, for the same level of prices).

13. We can rewrite $x = \frac{sB}{p_x}$ and $y = (1-s)\,B$. Then we can write an unconstrained maximization problem $\max_s u(s) = c\sqrt{\frac{sB}{p_x}} + \sqrt{(1-s)\,B}$.

14. The first order condition of this problem is

$$\begin{aligned}
0 &= \frac{c}{2}\sqrt{\frac{B}{p_x s^*}} - \frac{\sqrt{B}}{2\sqrt{(1-s^*)}} \\
c\sqrt{1-s^*} &= \sqrt{p_x s^*} \\
c^2\,(1-s^*) &= p_x s^* \\
\frac{c^2}{p_x + c^2} &= s^*
\end{aligned}$$

15. The second derivative $u(s)$ is

$$\begin{aligned}
\frac{d^2u}{ds^2} &= -\frac{c}{4}\sqrt{\frac{B}{p_x s^3}} - \frac{\sqrt{B}}{4\sqrt{(1-s)^3}} \\
&= -\frac{\sqrt{B}}{4}\left(\frac{c}{\sqrt{p_x s^3}} + \frac{1}{\sqrt{(1-s)^3}}\right)
\end{aligned}$$

Since the shares are always positive and less than one, the quantity in the parenthesis is positive and therefore $\frac{d^2M}{ds^2} < 0$

16. Applying the implicit function theorem

$$
\begin{aligned}
\frac{ds^*}{dc} &= \frac{-\sqrt{\frac{B}{4p_x s^*}}}{-\frac{\sqrt{B}}{4}\left(\frac{c}{\sqrt{p_x s^{*3}}}+\frac{1}{\sqrt{(1-s^*)^3}}\right)} \\
&= \frac{2/\sqrt{p_x\frac{c^2}{p_x+c^2}}}{\left(\frac{c}{\sqrt{p_x\left(\frac{c^2}{p_x+c^2}\right)^3}}+\frac{1}{\sqrt{\left(1-\frac{c^2}{p_x+c^2}\right)^3}}\right)} \\
&= \frac{2/\sqrt{p_x\frac{c^2}{p_x+c^2}}}{\frac{c}{\sqrt{p_x\left(\frac{c^2}{p_x+c^2}\right)^3}}+\frac{1}{\sqrt{\left(\frac{p_x}{p_x+c^2}\right)^3}}} \\
&= \frac{2/\sqrt{p_x c^2}\cdot\left(p_x+c^2\right)}{\frac{c}{\sqrt{p_x c^6}}+\frac{1}{\sqrt{p_x^3}}} \\
&= \frac{2\left(p_x+c^2\right)/c}{\frac{1}{c^2}+\frac{1}{p_x}} \\
&= \frac{2\left(p_x+c^2\right)}{\frac{1}{c}+\frac{c}{p_x}} \\
&= \frac{2\left(p_x+c^2\right)cp_x}{p_x+c^2} \\
&= 2cp_x
\end{aligned}
$$

PopcornCo's CEO wants c to go up. Skipping all the algebra, in the first line you can see the numerator *and* denominator is negative.

17. You can just use all the derivations that we had before since preferences are preserved under monotonic transformation.

$$
\begin{aligned}
v &= u(x,y)^2 \\
&= \left(c\sqrt{x}+\sqrt{y}\right)^2 \\
&= c^2x + y + 2c\sqrt{xy}
\end{aligned}
$$

Squaring the utility is a monotonic transformation, this utility is just the same as above except with $c = 2$. Therefore the Marshallian demand for popcorn is given by

$$
x^* = \frac{4}{4p_x + p_x^2}B
$$

2 Problem 2

1. Rational preferences are:

   (a) Complete, i.e. $\forall x, y$ $x \succeq y$ OR $y \succeq x$ (or both)

   (b) Transitive, i.e. if $x \succeq y$ and $y \succeq z$ then $x \succeq z$

2. $\succeq$ is complete. Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ be arbitrary sets of real numbers. By the completeness of the real numbers $x_i \geq y_i$ or $y_i \geq x_i$. There are a bunch of cases:

   (a) 3 x's greater: $x_i \geq y_i \quad \forall i$ Then $\mathbf{x} \succeq \mathbf{y}$.

   (b) 2x's greater: then $\mathbf{x} \succeq \mathbf{y}$

      i. $x_i \geq y_i \quad \forall i \in \{1, 2\}$ and $y_3 \geq x_3$

      ii. $x_i \geq y_i \quad \forall i \in \{2, 3\}$and $y_1 \geq x_1$

      iii. $x_i \geq y_i \quad \forall i \in \{1, 3\}$and $y_2 \geq x_2$

   (c) 1 x greater: then $\mathbf{y} \succeq \mathbf{x}$

      i. $x_1 \geq y_1$ and $x_i \leq y_i \quad \forall i \in \{2, 3\}$

      ii. $x_2 \geq y_2$ and $x_i \leq y_i \quad \forall i \in \{1, 3\}$

      iii. $x_3 \geq y_3$ and $x_i \leq y_i \quad \forall i \in \{1, 2\}$

   (d) 0 x greater: $x_i \leq y_i \quad \forall i$. Then $\mathbf{y} \succeq \mathbf{x}$

3. $\succeq$ is not transitive. Let $\mathbf{x} = (1, 5, 10)$, $\mathbf{y} = (2, 6, 1)$ and $\mathbf{z} = (3, 3, 4)$. One can see $\mathbf{y} \succeq \mathbf{x}$ and $\mathbf{z} \succeq \mathbf{y}$ and $\mathbf{z} \not\succeq \mathbf{x}$, which violates transitivity.