Astronomy 7A Midterm #1

September 29, 2016

Name: ______________________________

Section: ______________________________

There are 2 problems and 11 subproblems.

Write your answers on these sheets showing all of your work. It is better to show some work without an answer than to give an answer without any work. Please clearly label which work corresponds to which problem.

Calculators are allowed to perform arithmetic. Please turn off all cellphones.

If you have any questions while taking the midterm, get the attention of one of the GSIs.

Budget your time; you will have from 11:10 am to 12:30 pm to complete the exam. Of course, you are free to hand in your exam before 12:30 pm. Make sure that you have time to at least briefly think about every required question on the midterm.

You do not need to work on the questions in order, so it is OK to skip a question and come back to it later.

On my honor, I have neither given nor received any assistance in the taking of this exam

Signed: ______________________________

Constants and Some Userful Formulae

c = 3.00×10^{10} cm/s = 3.00×10^{8} m/s
h = 6.626×10^{-27} erg s = 6.626×10^{-34} J s
$\sigma_{\mathrm{SB}} = 5.67 \times 10^{-8}$ W m^{-2} K^{-4} = 5.67×10^{-5} erg s^{-1} cm^{-2} K^{-4}
$k_{\mathrm{B}} = 1.38 \times 10^{-23}$ J/K = 1.38×10^{-16} erg/K
$m_p = 1.673 \times 10^{-24}$ g = 1.673×10^{-27} kg
$m_e = 9.11 \times 10^{-28}$ g = 9.11×10^{-31} kg
$L_\odot = 3.90 \times 10^{26}$ W = 3.90×10^{33} erg s^{-1}
Solar mass: $M_\odot = 2.0 \times 10^{33}$ g = 2.0×10^{30} kg
$R_\odot = 7.0 \times 10^{10}$ cm = 7.0×10^{8} m
Absolute Magnitude of Sun: $M_\odot = +4.74$
$R_\oplus = 6.4 \times 10^{8}$ cm = 6.4×10^{6} m
1 AU = 1.5×10^{13} cm = 1.5×10^{11} m
1 pc = 3.09×10^{18} cm = 3.09×10^{16} m
1 radian = 206265 arcsec
1 year = 12 months = 365 days = 3.15×10^{7} s

Classical Doppler Shift ($v_r \ll c$):

$$\frac{(\lambda_{\mathrm{observed}} - \lambda_{\mathrm{rest}})}{\lambda_{\mathrm{rest}}} = \pm v_r/c$$

Apparent magnitudes difference:

$$m_1 - m_2 = -2.5 \log \frac{F_1}{F_2}$$

Distance modulus:

$$m - M = 5\log_{10}(d) - 5 = 5\log_{10}\left(\frac{d}{10\mathrm{pc}}\right)$$

Stefan Boltzmann equation (A=area):

$$L = A\sigma_{\mathrm{SB}} T^4$$

Photon energy:

$$E = \frac{hc}{\lambda}$$

Planck Function:

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/(\lambda k_{\rm B}T)} - 1}$$

Wien Peak Law for B_λ:

$$\lambda_{\rm peak}T = 2.8977721 \times 10^{-3}\ {\rm m\,K}$$

Diffraction (Rayleigh) Limit:

$$\theta_{\rm diff} = 1.22\,\frac{\lambda}{D}$$

Plate scale:

$$\frac{d\theta}{dy} = \frac{1}{f}$$

Energy Levels of Hydrogen (equivalent of Rydberg equation):

$$E_n = \frac{-13.606\ {\rm eV}}{n^2}$$

Optical Depth:

$$\tau = \int n\sigma dx = \int \rho\kappa dx$$

Attenuation of Flux:

$$F = F_0 \cdot e^{-\tau}$$

Column Density:

$$N = \int n dx$$

Mean Free Path:

$$\lambda_{\rm mfp} = 1/(n\sigma) = 1/(\rho\kappa)$$

1 GAIA [64 points]

GAIA is a space observatory from the European Space Agency launched on December 19, 2013. One of GAIA's major goals is to determine the position, parallax, and annual proper motion of 10s of millions of stars in the Milky Way. Hence, the observatory will yield the largest and most precise 3D space catalog ever made. GAIA had its first public data release only a few weeks ago, and is already changing our astronomical landscape.

GAIA has three instruments on board, the astrometry instrument, the photometric instrument, and the radial-velocity spectrometer.

(a) [8 points] The astrometric instrument collects light over a wavelength range 330-1050 nm. The collecting area of the primary mirror is $0.725\,\mathrm{m}^2$. Assume that the filter response is flat as a function of wavelength and that the mirror is circular. What is the approximate diffraction (Rayleigh) limit of GAIA (in arcseconds) when using the astrometric instrument?

If the collecting area is $0.725\,m^2$ and the mirror is circular, then the radius of the mirror is $r = \sqrt{\frac{A}{\pi}} = 0.48\,m$ and thus the diameter $D = 0.96$

The response is flat and thus the average wavelength of the filter is 690 nm.

We now have all information needed to derive the diffraction limit:

$$\theta_{\mathrm{diff}} = 1.22\,\frac{\lambda}{D} = 1.22\,\frac{690 \times 10^{-9}}{0.96} = 8.77 \times 10^{-7}\ \mathrm{radian} = 0.18\ \mathrm{arcsec}$$

The accuracy at which the position of an object can be determined is significantly better than the diffraction limit for isolated sources. This is due to the shape of the mirror (it's actually rectangular), the availability of two primary mirrors, the observing strategy, the stability of the instrument, and several other factors. For a bright isolated star with an apparent magnitude of 15, GAIA can determine its position with an accuracy of 20 micro-arcseconds.

(b) [8 points] Given the above accuracy of 20 micro-arcseconds, would it be possible to measure the parallax of a 15th magnitude star with GAIA that is 10 kpc away from Earth? You can assume that GAIA orbits the Sun with a radius of 1 AU.

A star at a distance of 10 kpc will have a parallax angle of

$$p'' = \frac{1}{d} = \frac{1}{10 \times 10^3} = 1 \times 10^{-4} \text{ arcsec} = 100 \ \mu\text{arcsec}$$

With d in pc. Thus, the maximum angular movement is 200 μarcsec. As this is about 10 times larger than the accuracy of 20 μarcsec, we can indeed measure the parallax of this star.

(c) [8 points] What is the absolute magnitude and luminosity of this star in solar luminosities?

We can use the distance modulus equation to derive the absolute magnitude. The distance modulus for 10 kpc is

$$m - M = 5\log_{10}\left(\frac{d}{10\text{pc}}\right) = 15$$

Thus for a star with an apparent magnitude $m = 15$*, the absolute magnitude* $M = 15 - 15 = 0$*.*

In order to derive the luminosity of the star, we can use the following equations, and the fact that the absolute magnitude of the Sun is $M_\odot = 4.74$*:*

$$M_* - M_\odot = -2.5\log_{10}\left(\frac{L_*}{L_\odot}\right)$$

Thus,

$$L_* = 10^{-(M_* - M_\odot)/2.5} L_\odot = 10^{-(0.-4.74)/2.5} L_\odot = 79\, L_\odot$$

(d) [8 points] GAIA will also measure tangential speeds by observing stars 70 times over a 5 year time period. Our 15th magnitude star has a tangential velocity of 5 km/s. What is the total angular distance (in micro-arcsec) by which the star will move across the sky over a period of exactly 5 years?

The tangential speed and proper motion are related following $v_\theta = \mu r$, with r the distance to the object. Thus, the proper motion of this star is

$$\mu = \frac{v_\theta}{r} = \frac{5 \times 10^3 \text{ m/s}}{10^4 \text{ pc} \times 3.09 \times 10^{16} \text{ m/pc}} = 1.62 \times 10^{-17} \text{ rad/s} = 105 \ \mu\text{arcsec/yr}$$

Thus, in 5 years, the star will move by 527μarcsec

(e) [12 points] The proper motion of the star is exactly perpendicular to the ecliptic. Make a sketch of the total movement of the star as seen through GAIA over a period of five years (Figure 1). Take the plane of the ecliptic to be parallel to the x-axis in Figure 1. Add values to the axes.

The proper motion of the star is exactly perpendicular to the ecliptic, which means that the star is approximately in the ecliptic plane. The proper motion will make the star move up or down by a constant speed, while the parallax movement will cause the star to move to the left and right in exactly perpendicular direction to the proper motion. Thus, the combined movement will be a sinusoidal movement, as shown in Figure 1.

Please note that a spiraling movement would have been observed if the star would have been much higher or lower than the ecliptic, as in this case the parallax would contribute in the vertical direction as well.

(f) [8 points] GAIA cannot only measure the tangential, but also the radial (i.e. line of sight) velocity using the Radial-Velocity Spectrometer (RVS). The total velocity of the star is 10 km/s, and the distance between the star and our Sun is decreasing over time. Given that the spectrometer looks for a spectral line with rest wavelength $\lambda_{rest} = 850$ nm, what doppler shift $\Delta\lambda$

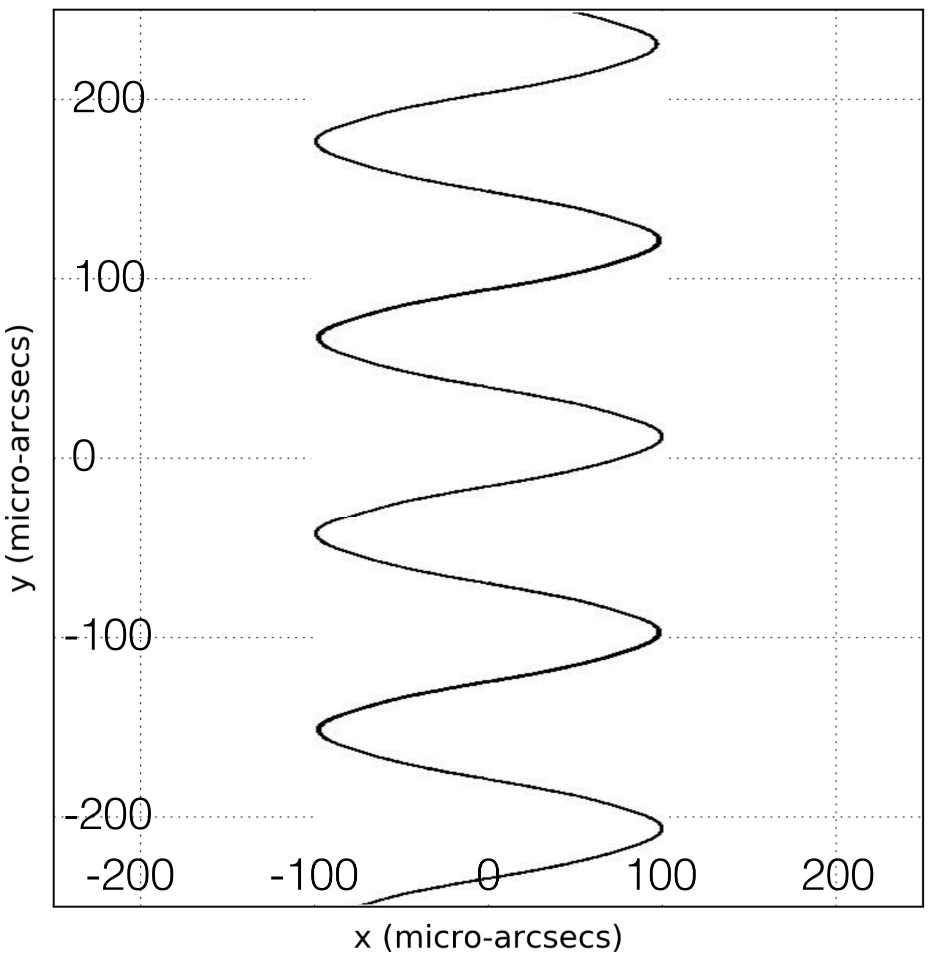

Figure 1: Trace the movement of the star as seen by GAIA over the course of five years.

will be detected for this star in nanometers? Also indicate whether the shift is toward bluer or redder wavelengths.

While we are given the total velocity of the star, it is only the velocity of the star along the line of sight which contributes to the doppler shift of the light. We determine this radial velocity by using the total velocity information given earlier in the problem.

$$v_T = \sqrt{v_R^2 + v_\theta^2}$$
$$v_R = \sqrt{v_T^2 - v_\theta^2}$$
$$v_R = \sqrt{(10 \text{ m/s})^2 - (5 \text{ m/s})^2} = 8660 \text{ m/s}$$

We can use this radial velocity directly in the doppler shift equation. We could either be sensitive to the sign, or reintroduce the direction after calculating the shift via magnitudes. We will choose the latter.

$$\frac{\Delta\lambda}{\lambda_{rest}} = \frac{v_R}{c}$$
$$\Delta\lambda = \frac{v_R}{c} \cdot \lambda_{rest}$$
$$\Delta\lambda = \frac{8660 \text{ m/s}}{3.00 \times 10^8 \text{ m/s}} \cdot 850. \times 10^{-9} \text{ m} = 0.0245 \text{ nm}$$

We are told in the problem that the distance between the star and the Sun is decreasing over time. This will result in blue shifted light. This also could have been found from using the appropriate negative sign for objects which are approaching the observer.

(g) [12 points] Now suppose the 15th magnitude star is actually obscured by a cloud of free electrons, which lies between the star and GAIA. The cloud has a column density of $N = 10^{24}$ cm^{-2}. The cross section, σ, of scattering between a photon and an electron is $\sigma = 6.65 \times 10^{-25}$ cm^2. What would the apparent magnitude of the star be if the cloud was not there? [Hint: You first need to solve for the optical depth of the cloud].

The optical depth is $\tau = N\sigma = 0.665$, which means it is non-negligible! We know the cloud will take flux away from the star, so a good check on our answer is that the star's intrinsic luminosity should be *brighter* that what we measure, or its apparent magnitude sans-cloud should be *less* than our measured apparent magnitude. We can calculate how much less using the flux attenuation equation. The flux ratio taken by the cloud is just $\frac{F}{F_0} = e^{-\tau} = 0.514$. This means the magnitude difference is $\Delta m = -2.5 \log_{10}(0.514) = 0.73$. This means $m_{app} = 15 - 0.73 = 14.27$ if the cloud wasn't there.

2 Wolf-Rayet Stars [36 points]

Wolf-Rayet stars are a type of massive stars with typical surface temperatures of 100,000 K. They are rare (only 200 are known in the Milky Way) and are characterized by strong emission lines in their spectra (in addition to absorption lines). These strong emission lines are the result of powerful stellar winds that eject material from the atmosphere into expanding spherical shells of hot gas surrounding the star. The speeds of the winds can be as high as 2000 km/s.

(a) [6 points] At what wavelength (in nanometers) does the spectrum (F_λ) of these stars peak?

We can use Wien's displacement law for B_λ:

$$\lambda_{\rm peak} T = 2.8977721 \times 10^{-3}\ {\rm m\,K}$$

Thus,

$$\lambda_{\rm peak} = \frac{2.8977721 \times 10^{-3}\ {\rm m\,K}}{T} = \frac{2.8977721 \times 10^{-3}\ {\rm m\,K}}{1 \times 10^{5}\ {\rm K}} = 2.9{\times}10^{-8}\,{\rm m} = 29\,{\rm nm}$$

(b) [10 points] Wolf-Rayet stars are also extremely bright, with their luminosity being approximately 1 million times higher than the Sun's luminosity. What is the typical radius of a Wolf-Rayet star in solar radii?

We can use the Stefan-Boltzmann relation to solve this problem:

$$L = A\sigma_{\rm SB}T^4 = 4\pi R^2 \sigma_{\rm SB} T^4$$

Thus,

$$R = \sqrt{\frac{L}{4\pi\sigma_{\rm SB}T^4}} = \sqrt{\frac{1 \times 10^6 L_\odot}{4\pi\sigma_{\rm SB}(1 \times 10^5)^4}} = 2.34 \times 10^9\ {\rm m} = 3.34\,R_\odot$$

(c) [8 points] Some Wolf-Rayet stars have hydrogen emission lines in their spectra. For these stars the Balmer α emission line at 656.3 nm is visible. What orbital transition is responsible for creating this line? In other words, give the initial and final principle quantum numbers n_{initial} and n_{final}. Check your answer using Rydberg's equation.

The Balmer α line is caused by a transition from the quantum numbers $n_{initial} = 3$ to $n_{final} = 2$. Note that this must be in this order, for emission only occurs when electrons travel from a higher to a lower orbital number. Going from $n_{initial} = 2$ to $n_{final} = 3$ would mean that the observed line is in absorption instead of emission. To check our answer using Ryberg's equation we calculate the difference in energy levels between these two quantum states.

$$\Delta E = -13.6 \text{ eV} \left(\frac{1}{n_{\text{final}}^2} - \frac{1}{n_{\text{initial}}^2} \right)$$

$$\Delta E = -13.6 \text{ eV} \frac{1.602 \times 10^{-19} \text{ J}}{1 \text{ eV}} \left(\frac{1}{2^2} - \frac{1}{3^2} \right) = -3.03 \times 10^{-19} \text{ J}$$

If we have chosen the correct principle quantum numbers, than this energy gap should be equal to the energy of the photon released by the transition. While the energy of the electron is decreased, this should transform into a positive energy for the photon.

$$E_\gamma = \frac{hc}{\lambda}$$

$$E_\gamma = \frac{(6.626 \times 10^{-34} \text{ J} \cdot \text{s}) \cdot (3.00 \times 10^8 \text{ m/s})}{656.3 \times 10^{-9} \text{ m}} = 3.03 \times 10^{-19} \text{ J}$$

Some students proved the correct principle quantum numbers had been chosen by using the energy found from Rydberg's equation to solve for the wavelength of the photon. Still yet others used the wavelength to find the energy and then attempted using different values in Rydberg's equation until showing that the correct answer was $n_{initial} = 3$ to $n_{final} = 2$. All of these techniques, if successfully compelted, were awarded full credit.

(d) [12 points] The Balmer α line originates from the expanding shell of hot gas surrounding the star. The star itself is optically thick and its spectrum is approximately flat in F_λ in the wavelengths surrounding the Balmer α rest wavelength. Sketch the integrated line profile of Balmer α, indicate the rest wavelength (assume it's at rest compared to the Sun), and label the approximate regions A-D from Figure 2 from where the light originates.

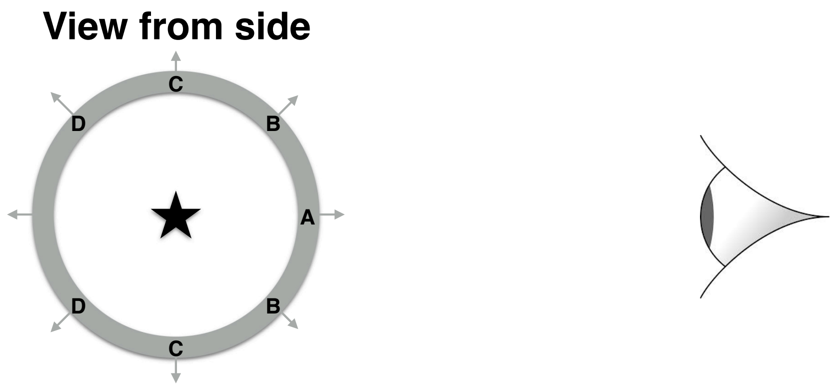

Figure 2: Wolf-Rayet star surrounded by expanding spherically symmetric shell. This image shows the *cross section* of the shell as seen from the side. The observer is to the right, as indicated by the eye.

This problem was just like the homework problem, but with the absorption line on the other side.

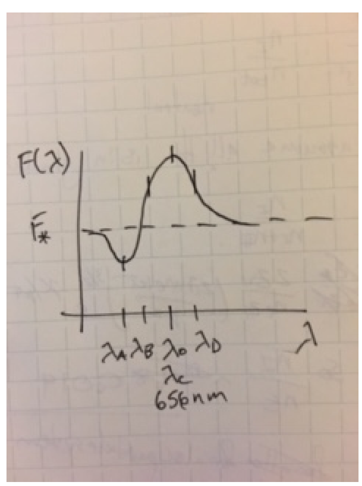

Figure 3: An answer that will get full credit.