Final Exam Solutions

Physics 7B
Lecture 3

Fall 2015

Midterm 3, Question 1 Solutions - Prepared by Gilbert Lopez

1.1

The correct choice is a (2 points).

1.2

The correct choice is b (3 points).

1.3

The correct choice is a (2 points). Volume is increasing, and so the work done by the gas is positive (1 point), which means a is the only possible choice. For incorrect answers, 1 point is given if students state that, by definition, $\Delta T=0$ for isothermal processes, which means $\Delta U=0$ (1 point). .

1.4

The correct choice is b (1 point). The field lines point in the wrong direction in a and d (1 point). In c and d, the field lines are discontinuous (or you can imply by the drawing that they intersect, which they shouldn't, or that c represents the field lines of individual point charges, and not their superposition) (1 point).

1.5

Greater (2 points). The slots limit where eddy currents can be produced, leading to a smaller effect (1 point).

PROBLEM 2

Part a.

- For A $\to$ B, by the ideal gas law, the final temperature is

$$T_B = \frac{3P_0V_0}{nR} = 3T_0. \tag{1}$$

Since the process is isochoric, the absorbed heat is

$$Q_1 = nC_v\Delta T = nR\frac{3}{2}(3T_0 - T_0) = 3nRT_0. \tag{2}$$

You get $\boxed{+2}$ points for finding this.

- For B $\to$ C, the final temperature is

$$T_C = \frac{6P_0V_0}{nR} = 6T_0. \tag{3}$$

Since the process is isobaric, the absorbed heat is

$$Q_2 = nC_P\Delta T = nR\frac{5}{2}(6T_0 - 3T_0) = \frac{15}{2}nRT_0. \tag{4}$$

You get $\boxed{+2}$ points for finding this. So the total heat entering is

$$Q_{\text{in}} = Q_1 + Q_2 = \frac{21}{2}nRT_0. \tag{5}$$

You get $\boxed{+1}$ point for correctly adding these heats.

Part b.

- For C $\to$ D, by the ideal gas law, the final temperature is

$$T_D = \frac{2P_0V_0}{nR} = 2T_0. \tag{6}$$

Since the process is isochoric, the absorbed heat is

$$Q_3 = nC_v\Delta T = nR\frac{3}{2}(2T_0 - 6T_0) = -6nRT_0. \tag{7}$$

You get $\boxed{+2}$ points for finding this.

- For D $\to$ A, the final temperature is T_0. Since the process is isobaric, the heat is

$$Q_4 = nC_P\Delta T = nR\frac{5}{2}(T_0 - 2T_0) = -\frac{5}{2}nRT_0. \tag{8}$$

You get $\boxed{+2}$ points for finding this.

The total heat leaving the system is the negative of the heats absorbed in C $\to$ D and D $\to$ A:

$$Q_{\text{out}} = -Q_3 - Q_4 = \frac{17}{2}nRT_0. \tag{9}$$

You get $\boxed{+1}$ point for correctly adding these heats.

Part c. The efficiency is given by

$$\eta = \frac{Q_{\text{in}} - Q_{\text{out}}}{Q_{\text{in}}} = \frac{4}{21}. \tag{10}$$

You get $\boxed{+2}$ points for correctly writing down this general formula for efficiency (or the corresponding version with works), and $\boxed{+3}$ points for correctly evaluating this fraction.

Part d. The highest temperature in the cycle is $T_C = 6T_0$, the lowest temperature is $T_A = T_0$. The efficiency of a Carnot engine between these temperatures is

$$\eta_{\text{Carnot}} = \frac{T_H - T_L}{T_H} = \frac{5}{6}, \tag{11}$$

which, as expected by the second law of thermodynamics, is higher than the efficiency of our engine. You get $\boxed{+2}$ points for remembering the formula for the Carnot efficiency, $\boxed{+2}$ points for finding T_H and T_L and $\boxed{+1}$ point for correctly evaluating this efficiency.

Problem 3

Consider a solid non-conducting sphere of radius a carrying a non-uniform positive charge density given by $\rho(r) = \rho_0(r^2/a^2)$, where ρ_0 is a positive constant. A very thin non-conducting concentric spherical shell of radius b, with $b > a$ carries a positive charge Q uniformly distributed on the surface. Determine the direction and magnitude of the electric field in each of the regions:

1. Region I: $(r < a)$
   We proceed using Gauss's Law choosing a sphere concentric with the system and with radius $r < a$ as our gaussian surface. The amount of charge enclosed by this gaussian is

   $$Q_{enc}(r \leq a) = \int \rho dV = \int_0^r \rho_0 \frac{r'^2}{a^2}(4\pi r'^2 dr') = \frac{4\pi\rho_0 r^5}{5a^2}.$$

   Note in the integrand we have substituted r' for r as we are already using the variable r to denote the radius of our gaussian sphere. With Q_{enc} in hand, we are now ready to apply Gauss's Law.

   $$\oint \mathbf{E} \cdot \mathbf{dA} = \frac{Q_{enc}}{\epsilon_0}$$
   $$E(4\pi r^2) = \frac{4\pi\rho_0 r^5}{5\epsilon_0 a^2}$$
   $$\mathbf{E} = \frac{\rho_0 r^3}{5\epsilon_0 a^2}\hat{\mathbf{r}}$$

   Rough rubric (7pts):

   (a) 1pt for recognizing this problem is most easily solved via Gauss's Law
   (b) 1pt for choosing a valid gaussian surface
   (c) 2pt for correctly (or nearly correctly) determining $Q_{enc}(r)$
   (d) 1pt for showing the LHS of Gauss's Law reduces as follows $\int \mathbf{E} \cdot \mathbf{dA} = E(4\pi r^2)$
   (e) 1pt for arriving at the correct magnitude for $\mathbf{E}$
   (f) 1 point for stating the electric field points in the radial direction

2. Region II: $(a < r < b)$
   In this region, we choose a sphere with radius $a < r < b$ as our gaussian surface. Here, the enclosed charge is

   $$Q_{enc}(a<r<b) = Q_{enc}(r=a) = \frac{4\pi\rho_0 a^3}{5}.$$

Again, we apply Gauss's law

$$\begin{aligned}
\oint \mathbf{E} \cdot \mathbf{dA} &= \frac{Q_{enc}}{\epsilon_0} \\
E(4\pi r^2) &= \frac{4\pi \rho_0 a^3}{5\epsilon_0} \\
\mathbf{E} &= \frac{\rho_0 a^3}{5\epsilon_0 r^2}\hat{\mathbf{r}}
\end{aligned}$$

Rough rubric (7pts):

(a) 1pt for recognizing this problem is most easily solved via Gauss's Law
(b) 1pt for choosing a valid gaussian surface
(c) 2.5pt for correctly determining $Q_{enc}(a{<}r{<}b) = Q_{enc}(r{=}a)$
(d) 1.5pt for showing the LHS of Gauss's Law reduces as follows $\int \mathbf{E} \cdot \mathbf{dA} = E(4\pi r^2)$
(e) 1 point for stating the electric field points in the radial direction

3. `Region III:` $(r > b)$
Lastly, to determine the electric for $r > b$, we choose to our gaussian sphere to have a radius $r > b$. In this case, the total charge enclosed is

$$Q_{enc}(r{>}b) = Q_{enc}(r{=}a) + Q = \frac{4\pi \rho_0 a^3}{5} + Q$$

Once again, we apply Gauss's law

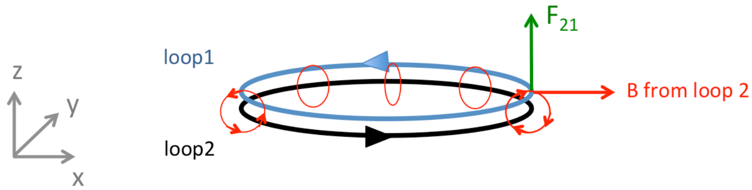
$$\begin{aligned}
\oint \mathbf{E} \cdot \mathbf{dA} &= \frac{Q_{enc}}{\epsilon_0} \\
E(4\pi r^2) &= \frac{4\pi \rho_0 a^3}{5\epsilon_0} + \frac{Q}{\epsilon_0} \\
\mathbf{E} &= \left(\frac{\rho_0 a^3}{5\epsilon_0 r^2} + \frac{Q}{4\pi\epsilon_0 r^2}\right)\hat{\mathbf{r}}
\end{aligned}$$

Rough rubric (6pts):

(a) 1.5pt for recognizing this problem is most easily solved via Gauss's Law AND choosing a valid gaussian surface
(b) 2pt for correctly determining $Q_{enc}(r{>}b) = Q_{enc}(r{=}a) + Q$
(c) 1.5pt for showing the LHS of Gauss's Law reduces as follows $\int \mathbf{E} \cdot \mathbf{dA} = E(4\pi r^2)$
(d) 1 point for stating the electric field points in the radial direction

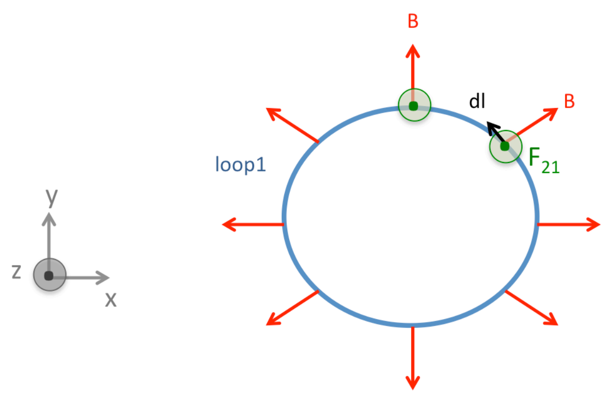
Problem 4.
Part a) (15 points total)

Near the bottom loop (loop 2), the magnetic field due to loop 2 looks like circles around the loop as shown below, and therefor the magnetic filed points radially outward anywhere on the top loop (loop 1) :



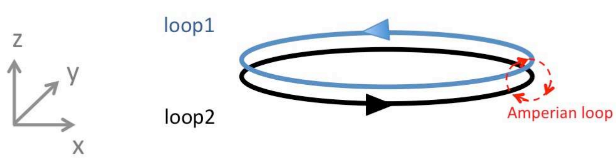
(2 point) to state or show what the magnetic field looks like near the bottom loop
(1 point) to state or show the direction of magnetic field on the top loop

Using the right hand rule and knowing that $dF = Idl \times B$, the direction of force on the loop is pointing upward (or in the $\hat{z}$) direction



(2 points) to state or show the direction of force on the top loop

Because the magnetic field lines are circular around the bottom loop, we can draw and amperian loop of radius d as shown below, to find the magnitude of magnetic field:



(2 points) to state or show how you found the magnitude of magnetic field, show or explain your amperian loop or the r vector if you used Biot Savart

using ampere's law:

$$\oint B.dl = \mu_0 I$$

(1 point) for writing down/using ampere's law or Biot Savart law

$$B = \frac{\mu_0 I}{2\pi d}$$

(3 points) correct value for magnitude of magnetic field

now we can find magnetic force on the top loop

$$dF = Idl \times B$$

(1 points) correct formula for force

$$F = I.2\pi.(10d).\frac{\mu_0 I}{2\pi d} = 10\mu_0 I^2$$

(3 points) correct value for magnitude of magnetic force

Part b) (5 points total)

$$\Sigma F = ma$$

(1 points) writing down second law of motion

$$ma = 10\mu_0 I^2(\hat{z}) - mg(\hat{z})$$

(1 points) writing down second law of motion in terms of given vectors

$$\vec{a} = \frac{10\mu_0 I^2}{m} - g(\hat{z})$$

(2 points) correct magnitude for acceleration (based on part a)

(1 points) correct direction for acceleration (based on part a)

Question 5

1 a

$$\Phi_B = \pi r^2 B(t=0) = \pi L^2 a$$

4 points for method
1 point for answer

deduce 2 points for wrong area, or wrong inital magnetic field

2 b

$$\xi = -\frac{d\Phi_B}{dt} = -\pi L^2 b$$

4 points for method
1 point for answer

answer for either sign accepted

3 c

$$I = \frac{\xi}{R} = -\frac{\pi L^2 b}{R}$$

4 points for method
1 point for answer

answer for either sign accepted

4 d

$$P = I^2 R = \frac{\pi^2 L^4 b^2}{R}$$

4 points for method
1 point for answer

6.

a) To solve this problem, we need to calculate the magnetic field by using the Biot-Savart Law, shown below:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{I\mathbf{dl}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\int \frac{I\mathbf{dl}\times\mathbf{r}}{r^3}$$

Since magnetic fields obey the principle of superposition, we can split this current distribution into several pieces, and calculate the **B**-field from each one. By the symmetry of the problem, it is clear that we should split the current loop into four pieces, corresponding to the sides of the square.

Let us consider just the top segment of the loop and determine its contribution. We choose a coordinate system with the origin O at the center of the square, the positive x-axis pointing to the right, the positive y-axis pointing up, and the positive z-axis coming out of the page. To calculate the Biot-Savart Law, we need to determine the values of the necessary parameters in a Cartesian system. I is constant. $\mathbf{dl}$ is a differential position element pointing in the direction of the current, so we have $\mathbf{dl} = dx\hat{\mathbf{x}}$. $\mathbf{r}$ is a vector that points from the current source to the observation point, so at some arbitrary point (x, y, z) on the current line, $\mathbf{r} = -x\hat{\mathbf{x}} - \frac{L}{2}\hat{\mathbf{y}}$. Finally, from this relationship, we have $r = \sqrt{x^2 + \frac{L^2}{4}}$. Now we can calculate the magnetic field:

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0 I}{4\pi}\int \frac{dx\hat{\mathbf{x}}\times(-x\hat{\mathbf{x}} - \frac{L}{2}\hat{\mathbf{y}})}{(x^2+\frac{L^2}{4})^{3/2}} \\
&= \frac{\mu_0 I}{4\pi}\int \frac{-xdx(\hat{\mathbf{x}}\times\hat{\mathbf{x}}) - \frac{L}{2}dx(\hat{\mathbf{x}}\times\hat{\mathbf{y}})}{(x^2+\frac{L^2}{4})^{3/2}} \\
&= \frac{\mu_0 I}{4\pi}\int \frac{-\frac{L}{2}dx(\hat{\mathbf{x}}\times\hat{\mathbf{y}})}{(x^2+\frac{L^2}{4})^{3/2}} \\
&= -\frac{\mu_0 IL}{8\pi}\hat{\mathbf{z}}\int \frac{dx}{(x^2+\frac{L^2}{4})^{3/2}}
\end{aligned}$$

Since the line of current goes from $x = -L/2$ to $x = L/2$, we can use these as our limits. Now, we must solve this integral:

$$
\begin{aligned}
B &\propto \int_{-L/2}^{L/2} \frac{dx}{(x^2+\frac{L^2}{4})^{3/2}} \\
&= \frac{8}{L^3}\int_{-L/2}^{L/2} \frac{dx}{(\frac{4x^2}{L^2}+1)^{3/2}} \\
&= \frac{8}{L^3}\int_{x=-L/2}^{x=L/2} \frac{\frac{L}{2}d(\frac{2x}{L})}{((\frac{2x}{L})^2+1)^{3/2}} \\
&= \frac{4}{L^2}\int_{x=-L/2}^{x=L/2} \frac{d(\frac{2x}{L})}{((\frac{2x}{L})^2+1)^{3/2}} \\
&= \frac{4}{L^2}\frac{\frac{2x}{L}}{\sqrt{1+(\frac{2x}{L})^2}}\Bigg|_{x=-L/2}^{x=L/2} \\
&= \frac{4\sqrt{2}}{L^2}
\end{aligned}
$$

So finally, we get that the contribution of the top wire is:

$$\mathbf{B} = \frac{\mu_0 I}{L}\frac{\sqrt{2}}{2\pi}(-\hat{\mathbf{z}})$$

It is easy to see from symmetry that each segment will contribute the same amount to the overall magnetic field. We get that at the center of the loop, the magnetic field is:

$$\mathbf{B} = \frac{\mu_0 I}{L}\frac{2\sqrt{2}}{\pi}(-\hat{\mathbf{z}})$$

+2 for Biot-Savart Law
+2 for splitting loop into 4 line segments
+1 for correct dl (integration variable)
+1 for correct r
+1 for correct integration limits
+2 for correct solution to the B-field of one segment
+1 for multiplying above answer by 4
+2 for correct direction of B-field

b) Again, we must use the Biot-Savart Law, but it is much easier. We again take the origin O to be at the center of the loop (which is now a circle). Assume that the circle has a radius of R and lies in the xy plane. As before, we shall say that the positive z-axis points out of the page, and that the current moves clockwise with magnitude I.

We can use cylindrical coordinates (r, θ, z) to simplify the problem. Then $\mathbf{dl} = Rd\theta(-\hat{\theta})$. We have chosen the negative direction because $\mathbf{dl}$ points in the direction of the current, which is clockwise, but $\hat{\theta}$ is defined to be counter-clockwise. And $\mathbf{r} = -R\hat{\mathbf{r}}$, from the current to the observation point. **Aside: This is terrible notation; the $\mathbf{r}$ refers to the vector in the Biot-Savart Law, while the $\hat{\mathbf{r}}$ is the unit radial vector in cylindrical coordinates.** Of course, this then means $r = R$. Now we can use the Biot-Savart Law to find the magnetic field:

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi} \int \frac{I\mathbf{dl} \times \mathbf{r}}{r^3} \\
&= \frac{\mu_0 I}{4\pi} \int \frac{R^2 d\theta(\hat{\theta} \times \hat{\mathbf{r}})}{R^3} \\
&= \frac{\mu_0 I}{4\pi} \frac{1}{R}(-\hat{\mathbf{z}}) \int d\theta
\end{aligned}$$

Of course, we integrate over the circle, so *theta* can range from 0 to 2π. Our answer thus is:

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0 I}{4\pi R}(-\hat{\mathbf{z}}) \int_0^{2\pi} d\theta \\
&= \frac{\mu_0 I}{4\pi R}(-\hat{\mathbf{z}})2\pi \\
&= \frac{\mu_0 I}{2R}(-\hat{\mathbf{z}})
\end{aligned}$$

But we are not done! We still need to relate this current loop to the original square loop. But that is easy, because they both have the same circumference: $2\pi R = 4L$. Thus $R = \frac{2L}{\pi}$. So we get the final answer:

$$\mathbf{B} = \frac{\mu_0 I}{L} \frac{\pi}{4}(-\hat{\mathbf{z}})$$

+1 for Biot-Savart Law
+1 for correct dl (integration element)
+1 for correct r (in B-S Law)
+1 for perpendicular relationship between dl and r
+1 for correct relationship between R and L
+2 for correct magnitude of B-field
+1 for correct direction of B-field

7.

a) The force that a magnetic field exerts on an electric current is given by:

$$\mathbf{F} = \int I\mathbf{dl} \times \mathbf{B}$$

Remember that in this equation, vector **dl** points in the same direction as the current, and we integrate over the entire current. In the diagram, we see that the current density **J** points downwards. **J** points in the same direction as the current. Moreover, the direction of **J** and **B** are constant. So at all points along the current, **dl** and **J** are parallel and constant. As a result, the force **F** points in the direction $\mathbf{J} \times \mathbf{B}$. Using the right-hand rule, we see that this is along the length of the pipe.

+4 for correct force equation (+2 for incorrect $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$)
+2 for showing that $\mathbf{F} \propto (\mathbf{J} \times \mathbf{B})$
+2 for showing that force points in direction of pipe

b) Remember that pressure is force divided by area: $P = F/A$. It is easy to calculate this quantity. As we showed before, the current and **B**-field are always perpendicular to each other. So we can calculate the force:

$$\mathbf{F} = \int I\mathbf{dl} \times \mathbf{B} = I(\int \mathbf{dl}) \times \mathbf{B} = I\mathbf{l} \times \mathbf{B}$$

$$F = IlB$$

Here, **l** is the full length that the current travels through. And using the normal notation, F, l, and B are the magnitudes of **F**, **l**, and **B**, respectively. Also, we do not have the current I directly, but we can find it. J is the current I divided by the cross sectional area of the current, so $I = JA = JwL$. Also from the diagram we can see that $l = h$. The net force that the current feels is thus:

$$F = JwLhB$$

Finally, the pressure on the liquid is the force divided by the cross-sectional area that the liquid travels through in the pipe:

$$P = F/A = F/(wh) = JLB$$

+2 for correct force equation
+2 for integrating properly to get $F = IlB$
+2 for getting current by $I = JA$
+1 for correct value of current I

+1 for correct length l
+2 for $P = F/A$
+1 for correct area in pressure equation
+1 for correct answer