Math 1B Final Exam, Solution

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The exam is closed book, apart from a sheet of notes 8"x11". Calculators are not allowed. It is your responsibility to write your answers clearly.

1. (6 points) Use substitution and integration by parts to find:

$$\int \tan x (\sec x)^2 \, e^{\tan x} dx$$

Solution: Substituting $s = \tan x$, $ds = (\sec x)^2 dx$ we have

$$\int \tan x (\sec x)^2 \, e^{\tan x} dx = \int s e^s ds.$$

Now we use integration by parts: $u = s$, $du = ds$, $dv = e^s ds$, $v = e^s$

$$\int s e^s ds = s e^s - \int e^s ds = s e^s - e^s + C.$$

Going back to the original variable:

$$\int \tan x (\sec x)^2 \, e^{\tan x} dx = \tan x \, e^{\tan x} - e^{\tan x} + C.$$

2. (6 points) Use trigonometric substitution to evaluate

$$\int \frac{1}{t^2\sqrt{1-t^2}} dt$$

for $t \in (0, 1)$.

Solution: Substitute $t = \sin\theta$, $0 < \theta < \frac{\pi}{2}$, so $dt = \cos\theta \, d\theta$ and $\sqrt{1-t^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$ since $0 < \theta < \frac{\pi}{2}$. Then

$$\int \frac{1}{t^2\sqrt{1-t^2}} dt = \int \frac{\cos\theta d\theta}{\sin^2\theta\cos\theta} = \int \frac{1}{\sin^2\theta} d\theta = \int \csc^2\theta \, d\theta = -\cot\theta + C.$$

We now go back to the original variable $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{1-t^2}}{t}$, then

$$\int \frac{1}{t^2\sqrt{1-t^2}} dt = -\frac{\sqrt{1-t^2}}{t} + C.$$

3. (7 points) First find the indefinite integral:

$$\int \frac{x}{x^2-3x-4}dx,$$

then, compute

$$\int_{-1}^{1} \frac{x}{x^2-3x-4}dx.$$

Is this a proper integral?

Solution: Using partial fractions, $x^2-3x-4=(x-4)(x+1)$, then

$$\frac{x}{x^2-3x-4}=\frac{x}{(x-4)(x+1)}=\frac{A}{x-4}+\frac{B}{x+1}.$$

This gives the equality $x=(A+B)x+(A-4B)$, then $A=\frac{4}{5}$ and $B=\frac{1}{5}$. Therefore

$$\int \frac{x}{x^2-3x-4}dx=\frac{4}{5}\int\frac{dx}{x-4}+\frac{1}{5}\int\frac{dx}{x+1}=\frac{4}{5}\ln|x-4|+\frac{1}{5}\ln|x+1|+C.$$

The integral $\int_{-1}^{1}\frac{x}{x^2-3x-4}dx$ is improper because of the lower limit of integration $x=-1$ where the function has a vertical asymptote. Using the integral we just computed

$$\begin{aligned}\int_{-1}^{1} \frac{x}{x^2-3x-4}dx &= \lim_{t\to-1^+}\int_{t}^{1}\frac{x}{x^2-3x-4}dx\\ &= \frac{4}{5}\ln 3+\frac{1}{5}\ln 2-\lim_{t\to-1^+}\Big(\frac{4}{5}\ln|t-4|+\frac{1}{5}\ln|t+1|\Big)\\ &= +\infty\end{aligned}$$

since $\lim\limits_{t\to-1^+}\ln|t+1|=-\infty$.

4. (5 points) Determine if the integral

$$\int_{1}^{\infty}\frac{x^2}{1+x^2}\,e^{-x}dx$$

converges or diverges using the comparison test.

Solution: The integrand is a nonnegative function so comparison test applies. Since $0\le\frac{x^2}{1+x^2}\le 1$ for all x and because $e^{-x}>0$ we have that for all x

$$0\le\frac{x^2}{1+x^2}\,e^{-x}\le e^{-x}.$$

Now $\int_1^\infty e^{-x}=\lim\limits_{t\to\infty}\int_1^t e^{-x}dx=\lim\limits_{t\to\infty}(-e^{-t}+e^{-1})=e^{-1}<\infty$. Since the function in the upper bound has a finite integral we conclude, by comparison test, that $\int_1^\infty\frac{x^2}{1+x^2}\,e^{-x}dx$ converges.

5. (7 points) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{(n^4+1)^2}$.

   i) Use the limit comparison test to show that the series is convergent.

   ii) Determine whether

$$0 \leq \sum_{n=1}^{\infty} \frac{n^3}{(n^4+1)^2} \leq \frac{3}{8}$$

   is true or false by comparing the series to an integral. (Hint: First find the interval where the integral comparison test is applicable. Then, use the integral to estimate the applicable portion of the series, and add the rest by hand.)

Solution: i) Limit compare to $\sum_{n=1}^{\infty} \frac{1}{n^5}$, letting $a_n = \frac{n^3}{(n^4+1)^2}$ and $b_n = \frac{1}{n^5}$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^3}{(n^4+1)^2}}{\frac{1}{n^5}} = \lim_{n\to\infty} \frac{n^8}{(n^4+1)^2} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n^4})^2} = 1 > 0.$$

Now $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges by p-test, $p = 5 > 1$, or by integral test. We conclude, by limit comparison test, that $\sum_{n=1}^{\infty} \frac{n^3}{(n^4+1)^2}$ converges.

ii) The lower bound is clearly true since the series has nonnegative terms only, so $0 \leq \sum_{n=1}^{\infty} \frac{n^3}{(n^4+1)^2}$. For the upper bound we compare to an integral. Let $f(x) = \frac{x^3}{(x^4+1)^2}$. Clearly f is nonnegative and continuous for $x \geq 1$. We need to know if f is decreasing and if so from where it starts to decrease.

$$f'(x) = \frac{-5x^6 + 3x^2}{(x^4+1)^3} = \frac{x^2(3-5x^4)}{(x^4+1)^3},$$

so $f'(x) < 0$ if and only if $|x| > (3/5)^{1/4}$, and in particular if $x \geq 1$. The estimate for the sum is

$$\sum_{n=1}^{\infty} \frac{n^3}{(n^4+1)^2} \leq \frac{1}{4} + \int_1^{\infty} \frac{x^3}{(x^4+1)^2} dx = \frac{1}{4} + \frac{-1}{4(x^4+1)}\Big|_1^{\infty} = \frac{3}{8}.$$

We have found

$$0 \leq \sum_{n=1}^{\infty} \frac{n^3}{(n^4+1)^2} \leq \frac{3}{8}$$

and so the statement in ii) is true.

6. (7 points) Determine if the series absolutely converges, conditionally converges, or diverges.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$$

Solution: The series is alternating because the sum $\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}$ can be written in the form $\sum_{n=1}^{\infty}(-1)^n b_n$ where $b_n = \frac{n}{n^2+1}$ is nonnegative. We apply the alternating series test. First the limit of $\{b_n\}$,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{n^2+1} = \lim_{n\to\infty}\frac{\frac{1}{n}}{1+\frac{1}{n^2}} = 0.$$

Now we check that $\{b_n\}$ is (eventually) decreasing. We can consider $\frac{d}{dx}\frac{x}{x^2+1} = \frac{1-x^2}{x^2+1} < 0$ if $x > 1$, and so the b_n's decrease. By alternating series test $\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}$ converges.

To see whether the series is absolutely convergent or conditionally convergent we need to study $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$. We can limit compare it to $\sum_{n=1}^{\infty}\frac{1}{n}$,

$$\lim_{n\to\infty}\frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}} = 1 > 0$$

and since the harmonic series diverges (p-test with $p = 1$ or integral test) we conclude that $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ diverges.

Thus $\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}$ is conditionally convergent.

7. (7 points) Find the radius and the interval of convergence of the power series.

$$\sum_{n=1}^{\infty}\frac{x^n}{2^n n^2}$$

Using the ratio test and calling $a_n = \frac{x^n}{2^n n^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}|x|\frac{2^n n^2}{2^{n+1}(n+1)^2} = \lim_{n\to\infty}\frac{|x|}{2}\left(\frac{n}{n+1}\right)^2 = \frac{|x|}{2},$$

so the series converges if $|x|/2 < 1$ and diverges if $|x|/2 > 1$. The radius of convergence is $R = 2$.

We know the series converges in the interval $(-2, 2)$. We now look at the endpoints. For $x = 2$

$$\sum_{n=1}^{\infty}\frac{2^n}{2^n n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2},$$

which converges by p-test with $p = 2 > 1$, or by integral test.

For $x = -2$

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{2^n n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2},$$

which converges because it is absolutely convergent, or by an application of the alternating series test.

The interval of convergence of the power series is $I = [-2, 2]$.

8. (7 points) Let $g(x) = \dfrac{x}{4+x}$

   i) Find the Taylor series expansion of $g(x)$ centered at $x = 1$.

   ii) Find $g^{(21)}(1)$, the 21^{st} derivative of g at $x = 1$.

   Solution: i) First note that $g(x) = \frac{x}{4+x} = \frac{x+4-4}{4+x} = 1 - \frac{4}{4+x}$ and so

$$g(x) = 1 - \frac{4}{4+x} = 1 - \frac{4}{5+(x-1)} = 1 - \frac{4}{5}\frac{1}{1+\frac{1}{5}(x-1)} = 1 - \frac{4}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{5^n}(x-1)^n,$$

   and the expansion is valid for $|\frac{1}{5}(x-1)| < 1$, that is for $-4 < x < 6$.

   ii) The coefficient of $(x-1)^{21}$ in the power series expansion of g around $x = 1$ is given by $\frac{g^{(21)}(1)}{21!}$. Using i) we find

$$g^{(21)}(1) = \frac{4}{5^{22}}21!$$

9. (7 points) Consider

$$f(x) = e^{2x} - e^{-2x}$$

   i) Find the Taylor series around $x = 0$. What is the coefficient of x^n?

   ii) What is its radius of convergence?

   iii) Using the power series obtained in i) compute $\lim_{x\to 0}(f(x)/x)$

   Solution: i) We can use the Taylor series of the exponential to find the Taylor series of f

$$e^{2x} = \sum_{n=0}^{\infty}\frac{(2x)^n}{n!} = \sum_{n=0}^{\infty}\frac{2^n x^n}{n!}$$
$$e^{-2x} = \sum_{n=0}^{\infty}\frac{(-2x)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^n x^n}{n!}$$

   and both power series have radius of convergence $R = \infty$. Then the Taylor series of f is

$$\sum_{n=0}^{\infty}\frac{2^n x^n}{n!} - \sum_{n=0}^{\infty}\frac{(-1)^n 2^n x^n}{n!} = \sum_{n=0}^{\infty}(1-(-1)^n)\frac{2^n}{n!}x^n,$$

   that can be rewritten as

$$\sum_{n=0}^{\infty}\frac{2^{2n+2}}{(2n+1)!}x^{2n+1} \tag{1}$$

   where we used that $1-(-1)^n$ equals 0 if n is even and equals 2 is n is odd.

   The coefficient of x^n is $(1-(-1)^n)\frac{2^n}{n!}$ or equivalently, it is 0 if n is even and $\frac{2^{n+1}}{n!}$ if n is odd.

ii) The radius of convergence is $R = \infty$ since it is obtained as the sum of two power series with infinite radius of convergence, and the radius of convergence of the sum of power series is at least the smallest between the two, which in this case is infinity.

We can also take the expression in (1) and apply the ratio test.

iii) From i) we have

$$f(x) = \sum_{n=0}^{\infty} \frac{2^{2n+2}}{(2n+1)!} x^{2n+1} = 4x + \sum_{n=1}^{\infty} \frac{2^{2n+2}}{(2n+1)!} x^{2n+1},$$

and

$$\frac{f(x)}{x} = 4 + \sum_{n=1}^{\infty} \frac{2^{2n+2}}{(2n+1)!} x^{2n}.$$

Since $\sum_{n=1}^{\infty} \frac{2^{2n+2}}{(2n+1)!} x^{2n} \to 0$ as $x \to 0$ we get $\lim_{x\to 0} f(x)/x = 4$.

10. (8 points) Consider the differential equation

$$\frac{dy}{dx} = \frac{y^2}{(x+1)^2}$$

i) Solve the differential equation. Express y explicitly in terms of x.

ii) Find a solution through $x = 1$, $y = 1$.

iii) Find the orthogonal trajectory through $x = 1$, $y = 1$.

Solution: i) Using the method for separable equations we get

$$\int \frac{dy}{y^2} = \int \frac{dx}{(x+1)^2}, \quad -\frac{1}{y} = -\frac{1}{x+1} + C, \quad y(x) = \frac{x+1}{1 - C(x+1)}.$$

ii) If $y(1) = 1$, we obtain $C = -\frac{1}{2}$, so the solution is

$$y(x) = \frac{2(x+1)}{3+x}.$$

iii) To find the orthogonal trajectories we need to solve the differential equation

$$\frac{dy}{dx} = -\frac{(x+1)^2}{y^2}$$

that is again separable.

$$\int y^2 dy = -\int (x+1)^2 dx, \quad \frac{y^3}{3} = -\frac{(x+1)^3}{3} + C, \quad y(x) = (3C - (x+1)^3)^{1/3}.$$

For the one passing through $x = 1$, $y = 1$ we obtain $C = 3$, then

$$y(x) = (9 - (x+1)^3)^{1/3}.$$

11. (7 points) Solve the equation

$$x\frac{dp}{dx} = p + x^2e^x \quad (x > 0)$$

and find a solution that satisfies $p(1) = 1$.

Solution: We rewrite the equation as

$$p' - \frac{1}{x}p = xe^x.$$

The integrating factor is $e^{\int -\frac{dx}{x}} = e^{-\ln|x|} = e^{\ln\frac{1}{|x|}} = \frac{1}{|x|} = \frac{1}{x}$ since $x > 0$. Following the method of the integrating factor,

$$\left(\frac{1}{x}p\right)' = e^x, \quad \frac{1}{x}p = \int e^x dx = e^x + C, \quad p(x) = xe^x + Cx.$$

Now if $p(1) = 1$ we need $C = 1 - e$ and

$$p(x) = xe^x + (1-e)x.$$

12. (10 points) Find the general solution to the differential equation

$$y'' + 2y' + y = x^2 + x + 1$$

and find the solution to the boundary value problem $y(0) = 1$, $y(1) = 2$.

Solution: We start with the homogeneous equation $y'' + 2y' + y = 0$ with characteristic equation $r^2+2r+1 = 0$, so $r = -1$ is the only solution. The solution to the homogeneous equation is

$$y_c = C_1e^{-x} + C_2xe^{-x}.$$

We use undetermined coefficients to find a particular solution. Set $y_p = Ax^2 + Bx + C$, so $y_p' = 2Ax + B$, $y_p'' = 2A$. Plugging it into the equation gives

$$Ax^2 + (4A + B)x + (2A + 2B + C) = x^2 + x + 1$$

from where $A = 1$, $4A + B = 1$, $2A + 2B + C = 1$. This gives $A = 1$, $B = -3$, $C = 5$ and the particular solution is

$$y_p(x) = x^2 - 3x + 5.$$

The general solution is $\quad y = C_1e^x + C_2xe^x + x^2 - 3x + 5.$

For the boundary value problem, $y(0) = C_1 + 5 = 1$ and $y(1) = C_1e^{-1} + C_2e^{-1} + 3 = 2$ that gives $C_1 = -4$ and $C_2 = 4 - e$. The solution is

$$y(x) = -4e^{-x} + (4-e)xe^{-x} + x^2 - 3x + 5.$$

13. (5 points) For the differential equation $y'' + y = \sec x$

    i) Show that $y = \cos(x)\ \ln|\cos x| + x\sin x$ is a solution to the given differential equation.

    ii) Find the general solution to the equation.

Solution: i)The derivatives are

$$\frac{d^2}{dx^2}\cos(x)\ \ln|\cos x| = -\cos x + \frac{\sin^2 x}{\cos x} - \cos(x)\ln|\cos x|,$$
$$\frac{d^2}{dx^2}x\sin x = 2\cos x - x\sin x.$$

We see that $y'' + y = \frac{\sin^2 x}{\cos x} + \cos x = \frac{1}{\cos x} = \sec x$.

ii) The general solution is given by the sum of the solution to the homogeneous equation and a particular solution. The characteristic equation of the homogeneous equation is $r^2 + 1 = 0$ with roots $r_1 = i, r_2 = -i$, so $y_c = C_1\cos x + C_2\sin x$. In i) we are given a particular solution so the general solution is

$$y = C_1\cos x + C_2\sin x + \cos(x)\ \ln|\cos x| + x\sin x.$$

14. (11 points) Follow the steps below to find the general solution to the given equation by using the power series method
$$y'' + 2xy' - y = 0$$

    i) Write power series for y, y' and y'' and find the recurrence for the coefficients.

    ii) Use the first part to write c_2, c_4 and c_6 in terms of c_0 and c_3, c_5 and c_7 in terms of c_1. Here c_2, c_3, etc are the coefficients in the power series expansion of y.

    iii) Use the previous part to write a general formula for the coefficients. Hint: even coefficients and odd coefficients will have a slightly different formula, so write two separate formulas for each case, even and odd.

    iv) Write the general solution to the equation.

    v) Find the radius of convergence of the solution. (Hint: Find the radii of convergence of any two linearly independent solutions, and take the smallest of the two.),

    vi) Find the solution to the initial value problem $y(0) = 1, y'(0) = 1$.

Solution: i) Setting $y = \sum_{n=0}^{\infty} c_n x^n$ we have

$$y' = \sum_{n=0}^{\infty} nc_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n.$$

Plugging into the equation gives

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + 2x\sum_{n=0}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

or equivalently

$$\sum_{n=0}^{\infty}((n+2)(n+1)c_{n+2} + 2nc_n - c_n)x^n = 0.$$

From here we obtain the recurrence

$$c_{n+2} = -\frac{2n-1}{(n+2)(n+1)}c_n, \text{ for } n \geq 0.$$

ii) Using i) we get

$$\begin{aligned} c_2 &= \frac{1}{2}c_0, \quad c_4 = -\frac{3}{3\cdot 4}c_2 = -\frac{3}{4!}c_0, \quad c_6 = -\frac{7}{5\cdot 6}c_4 = \frac{3\cdot 7}{6!}c_0 \\ c_3 &= -\frac{1}{2\cdot 3}c_1, \quad c_5 = -\frac{5}{4\cdot 5}c_3 = \frac{5}{5!}c_1, \quad c_7 = -\frac{9}{8\cdot 9}c_5 = -\frac{5\cdot 9}{9!}c_1. \end{aligned}$$

iii) We find that in general

$$\begin{aligned} c_{2n} &= \frac{(-1)^{n-1}\, 3\cdot 7\cdot 11\cdots(4n-5)}{(2n)!}c_0, \text{ for } n \geq 2, \\ c_2 &= \frac{1}{2}c_0, \\ c_{2n+1} &= \frac{(-1)^{n}\, 1\cdot 5\cdot 9\cdots(4n-3)}{(2n+1)!}c_1, \text{ for } n \geq 1. \end{aligned}$$

iv) The general solution is

$$\begin{aligned} y =& c_0\Big(1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}\, 3\cdot 7\cdot 11\cdots(4n-5)}{(2n)!}x^{2n}\Big) \\ &+ c_1\Big(x + \sum_{n=1}^{\infty}\frac{(-1)^{n}\, 1\cdot 5\cdot 9\cdots(4n-3)}{(2n+1)!}x^{2n+1}\Big). \end{aligned}$$

v) The radius of convergence of the first linearly independent solution (the one multiplied by c_0).

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= |x^2|\lim_{n\to\infty}\frac{4n-1}{(2n+1)(2n+2)} \\ &= |x^2|\lim_{n\to\infty}\frac{4/n-1/n^2}{(2+1/n)(2+2/n)} \\ &= 0, \end{aligned}$$

Then the radius of convergence is $R = \infty$. For the second linearly independent solution

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x^2|\lim_{n\to\infty}\frac{4n+1}{(2n+2)(2n+3)} = 0,$$

and thus the radius of convergence is $R = \infty$. Then the radius of convergence of the solution in iv) is $R = \infty$.

vi) Evaluating $y(0) = c_0 = 1$ and $y'(0) = c_1 = 1$ and the solution is

$$\begin{aligned} y =& 1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, 3 \cdot 7 \cdot 11 \cdots (4n-5)}{(2n)!} x^{2n} + \\ &+ x + \sum_{n=1}^{\infty} \frac{(-1)^{n}\, 1 \cdot 5 \cdot 9 \cdots (4n-3)}{(2n+1)!} x^{2n+1}. \end{aligned}$$