Math 1B Final Exam

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Lecture 1, Spring 2011

The exam is closed book, apart from a sheet of notes 8"x11". Calculators are not allowed. It is your responsibility to write your answers clearly. There are THREE PAGES of problems.

1. (6 points) Use substitution and integration by parts to find:

$$\int \tan x(\sec x)^2 \, e^{\tan x} dx$$

2. (6 points) Use trigonometric substitution to find the indefinite integral

$$\int \frac{1}{t^2\sqrt{1-t^2}} dt$$

assuming t is inside the interval, $t \in (0,1)$.

3. (7 points) First find the indefinite integral:

$$\int \frac{x}{x^2-3x-4} dx,$$

then, compute

$$\int_{-1}^{1} \frac{x}{x^2-3x-4} dx.$$

Is this a proper integral?

4. (5 points) Determine if the integral

$$\int_1^\infty \frac{x^2}{1+x^2} \, e^{-x} dx$$

converges or diverges using the comparison test.

5. (7 points) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{(n^4+1)^2}$.

   i) Use the limit comparison test to show that the series is convergent.

ii) Determine whether

$$0 \leq \sum_{n=1}^{\infty} \frac{n^3}{(n^4+1)^2} \leq \frac{3}{8}$$

is true or false by comparing the series to an integral. (Hint: First find the interval where the integral comparison test is applicable. Then, use the integral to estimate the applicable portion of the series, and add the rest by hand.)

6. (7 points) Determine if the series absolutely converges, conditionally converges, or diverges.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$$

7. (7 points) Find the radius and the interval of convergence of the power series.

$$\sum_{n=1}^{\infty} \frac{x^n}{2^n n^2}$$

8. (7 points) Let $g(x) = \dfrac{x}{4+x}$

   i) Find the Taylor series expansion of $g(x)$ centered at $x = 1$.

   ii) Find $g^{(21)}(1)$, the 21^{st} derivative of g at $x = 1$.

9. (7 points) Consider

$$f(x) = e^{2x} - e^{-2x}$$

   i) Find the Taylor series around $x = 0$. What is the coefficient of x^n?

   ii) What is its radius of convergence?

   iii) Using the power series obtained in i) compute $\lim_{x \to 0}(f(x)/x)$

10. (8 points) Consider the differential equation

$$\frac{dy}{dx} = \frac{y^2}{(x+1)^2}$$

   i) Solve the differential equation. Express y explicitly in terms of x.

   ii) Find a solution through $x = 1$, $y = 1$.

   iii) Find the orthogonal trajectory through $x = 1$, $y = 1$.

11. (7 points) Solve the equation

$$x\frac{dp}{dx} = p + x^2e^x \quad (x > 0)$$

and find a solution that satisfies $p(1) = 1$.

12. (10 points) Find the general solution to the differential equation

$$y'' + 2y' + y = x^2 + x + 1$$

and find the solution to the boundary value problem $y(0) = 1$, $y(1) = 2$.

13. (5 points) For the differential equation $y'' + y = \sec x$

    i) Show that $y = \cos(x)\ln|\cos x| + x\sin x$ is a solution to the given differential equation.

    ii) Find the general solution to the equation.

14. (11 points) Follow the steps below to find the general solution to the given equation by using the power series method

$$y'' + 2xy' - y = 0$$

   i) Write power series for y, y' and y'' and find the recurrence for the coefficients.

   ii) Use the first part to write c_2, c_4 and c_6 in terms of c_0 and c_3, c_5 and c_7 in terms of c_1. Here c_2, c_3, etc are the coefficients in the power series expansion of y.

   iii) Use the previous part to write a general formula for the coefficients. Hint: even coefficients and odd coefficients will have a slightly different formula, so write two separate formulas for each case, even and odd.

   iv) Write the general solution to the equation.

   v) Find the radius of convergence of the solution. (Hint: Find the radii of convergence of any two linearly independent solutions, and take the smallest of the two.),

   vi) Find the solution to the initial value problem $y(0) = 1, y'(0) = 1$.