First Midterm Examination
Tuesday September 27 2016
Closed Books and Closed Notes
Answer Both Questions

Question 1
A Particle on a Moving Plane Curve
(25 Points)

As shown in Figure 1, a particle of mass m is attached to a fixed point O by a linear spring of stiffness K and unstretched length $\ell_0 = 0$. The particle is free to move on a plane curve which is given a vertical oscillation $A\cos(\omega t)$. A vertical gravitational force $-mg\mathbf{E}_2$ also acts on the particle.

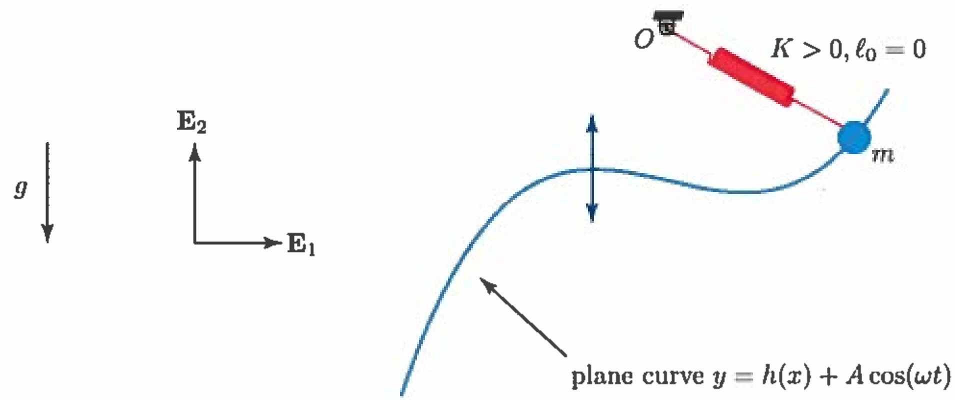


Figure 1: *Schematic of a particle of mass m which is attached to a fixed point O by a spring and is free to move on a plane space curve. A vertical gravitational force $-mg\mathbf{E}_2$ acts on the particle.*

The following coordinate system and its associated covariant basis vectors are used to describe the motion of the particle:

$$q^1 = x, \qquad q^2 = y - h(x), \qquad q^3 = z. \tag{1}$$

(a) *(5 Points)* What are the covariant basis vectors associated with this coordinate system?

(b) *(5 Points)* What are the contravariant basis vectors associated with this coordinate system?

(c) *(5 Points)* What are the constraints on the motion of the particle? Give a prescription for, and a physical interpretation of, the constraint force enforcing this constraint. In your solution assume that the curve is smooth.

(d) *(5 Points)* What are $\tilde{T}$ and $\tilde{U}$ of the particle? Why isn't $\tilde{T} + \tilde{U}$ conserved during a motion of the particle?

(e) *(5 Points)* Show that the equation of motion of the particle is

$$m\left(1 + h'h'\right)\ddot{x} + mh'h''\dot{x}^2 = -mh'\left(g - A\omega^2\cos(\omega t)\right) - K? \tag{2}$$

where $h' = \frac{\partial h}{\partial x}$ and $h'' = \frac{\partial^2 h}{\partial x^2}$. For full credit supply the missing term.

Question 2

A Particle on a Smooth Surface of Revolution (25 Points)

As shown in Figure 2, a particle of mass m is free to move on a rotating surface of revolution $z = f(r)$. The surface is rotating about a vertical axis with a speed $\Omega = \Omega(t)$.

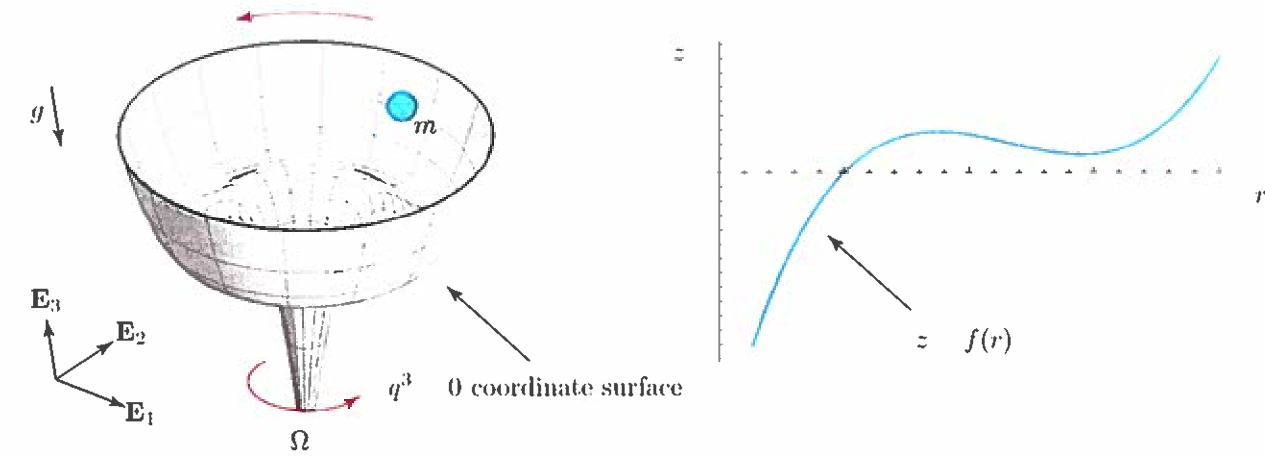


Figure 2: *Schematic of a particle of mass m which is moving on a surface of revolution in $\mathbb{E}^3$ under the influence of a gravitational force $-mg\mathbf{E}_3$.*

To establish the equations of motion for the particle, the following curvilinear coordinate system is defined for $\mathbb{E}^3$:

$$q^1 = r, \qquad q^2 = \theta, \qquad q^3 = \eta = z - f(r). \tag{3}$$

In your answers, please make use of the results on cylindrical polar coordinates on Page 3.

(a) *(5 Points)* Show that the covariant basis vectors $\mathbf{a}_k$ for this coordinate system are

$$\mathbf{a}_1 = \mathbf{e}_r + f'\mathbf{E}_3, \qquad \mathbf{a}_2 = r\mathbf{e}_\theta, \qquad \mathbf{a}_3 = \mathbf{E}_3, \tag{4}$$

where $f' = \frac{\partial f}{\partial r}$. Compute the matrix $[a_{ik}]$.

(b) *(5 Points)* Show that the contravariant basis vectors for this system are

$$\mathbf{a}^1 = ?, \qquad \mathbf{a}^2 = \frac{1}{r}\mathbf{e}_\theta, \qquad \mathbf{a}^3 = ??. \tag{5}$$

For full credit, supply the missing terms.

(c) *(10 Points)* Suppose that the particle is moving on the smooth surface of revolution $z = f(r)$ under a gravitational force $-mg\mathbf{E}_3$. Show that the differential equations of motion are

$$m\left(1 + f'f'\right)\ddot{r} - mr\dot{\theta}^2 + ??? = -mgf', \qquad \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0. \tag{6}$$

For full credit, supply the missing terms.

(d) *(5 Points)* Answer either (i) or (ii):

(i) Show that the total energy E and the angular momentum $\mathbf{H}_O \cdot \mathbf{E}_3$ of the particle are conserved during its motion on the surface of revolution.

(ii) For the specific $f(r)$ shown in Figure 2, what are the equilibria of the equations (6) and why are there infinitely many of them?

Notes on Cylindrical Polar Coordinates

Recall that the cylindrical polar coordinates $\{r, \theta, z\}$ are defined using Cartesian coordinates $\{x = x_1, y = x_2, z = x_3\}$ by the following relations:

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan\left(\frac{x_2}{x_1}\right), \quad z = x_3.$$

In addition, it is convenient to define the following orthonormal basis vectors:

$$\begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{E}_3 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}.$$

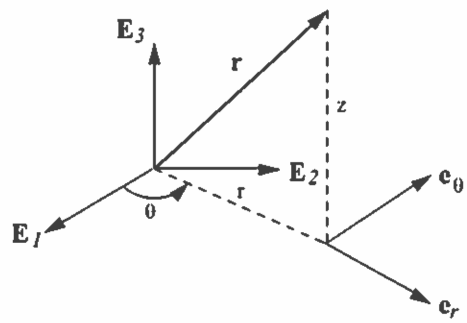


Figure 3: Cylindrical polar coordinates

For the coordinate system $\{r, \theta, z\}$, the covariant basis vectors are

$$\mathbf{a}_1 = \mathbf{e}_r, \quad \mathbf{a}_2 = r\mathbf{e}_\theta, \quad \mathbf{a}_3 = \mathbf{E}_3.$$

In addition, the contravariant basis vectors are

$$\mathbf{a}^1 = \mathbf{e}_r, \quad \mathbf{a}^2 = \frac{1}{r}\mathbf{e}_\theta, \quad \mathbf{a}^3 = \mathbf{E}_3.$$

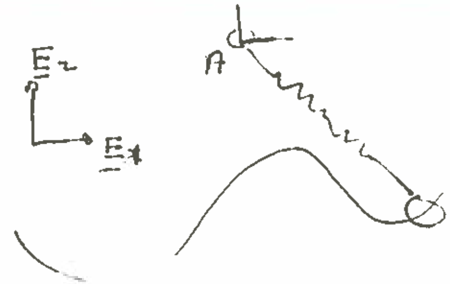
The gradient of a function $u(r, \theta, z)$ has the representation

$$\nabla u = \frac{\partial u}{\partial r}\mathbf{e}_r + \frac{\partial u}{\partial \theta}\frac{1}{r}\mathbf{e}_\theta + \frac{\partial u}{\partial z}\mathbf{E}_3.$$

For a particle of mass m which is unconstrained, the linear momentum $\mathbf{G}$, angular momentum $\mathbf{H}_O$, and kinetic energy T of the particle have the representations

$$\begin{aligned} \mathbf{G} &= m\dot{r}\mathbf{a}_1 + m\dot{\theta}\mathbf{a}_2 + m\dot{z}\mathbf{a}_3, \\ \mathbf{H}_O &= -mrz\dot{\theta}\mathbf{e}_r + m\left(z\dot{r} - r\dot{z}\right)\mathbf{e}_\theta + mr^2\dot{\theta}\mathbf{E}_3, \\ T &= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right). \end{aligned}$$

QUESTION 1



$$q^1 = x$$
$$q^2 = y - h(x)$$
$$q^3 = z$$

$$\underline{r} = q^1\underline{E}_1 + (q^2 - h(q^1))\underline{E}_2 + q^3\underline{E}_3$$

(a)

$$\underline{a}_1 = \underline{E}_1 + h'\underline{E}_2, \quad \underline{a}_2 = \underline{E}_2, \quad \underline{a}_3 = \underline{E}_3$$

(b) $\underline{a}^1 = \nabla q^1 = \underline{E}_1$

$$\underline{a}^2 = \nabla q^2 = \underline{E}_2 - h'\underline{E}_1$$
$$\underline{a}^3 = \nabla q^3 = \underline{E}_3$$

(c) Constraints $q^3 = 0$ and $q^2 - A\cos\omega t = 0$.

$$\underline{F}_c = \lambda_1 \underline{a}^2 + \lambda_2 \underline{a}^3$$

$\underline{F}_c$ = normal force act. on particle.

(d) $$\tilde{T} = \tfrac{1}{2} m(\dot{x}\underline{a}_1 + A\omega\sin\omega t\,\underline{a}_2)\cdot(\dot{x}\underline{a}_1 + A\omega\sin\omega t\,\underline{a}_2)$$

$$= \tfrac{1}{2} m\left((1+h'h')\dot{x}^2 + A^2\omega^2\sin^2\omega t - 2h'A\omega\sin\omega t\,\dot{x}\right)$$

$$\tilde{u} = (mgy = mg(h(x) + A\cos\omega t)) + \tfrac{1}{2}K(x^2 + (A\cos\omega t + h)^2)$$

$$\dot{E} = \dot{\tilde{T}} + \dot{\tilde{u}} = \underline{F}_{nc}\cdot\underline{v} = (\lambda_1\underline{a}^2 + \lambda_2\underline{a}^3)\cdot\underline{v} = \lambda_1(-A\sin\omega t\,\omega)$$

$$\neq 0.$$

(e) Equation of motion.

$$\frac{d}{dt}\left(\frac{\partial\tilde{T}}{\partial\dot{x}} = m(1+h'h')\dot{x} - h'mA\omega\sin\omega t\right)$$

$$-\left(\frac{\partial\tilde{T}}{\partial x} = mh'h''\dot{x}^2 - mh''A\omega\dot{x}\sin\omega t\right)$$

$$= -\frac{\partial\tilde{u}}{\partial x} = -mgh' - K(x + (A\cos\omega t + h)h')$$

Expand the $\frac{d}{dt}$ term

$$m(1+h'h')\ddot{x} + 2mh'h''\dot{x}^2 - mA\omega \sin\omega t\, h''\dot{x} - mA\omega^2\cos\omega t\, h'$$

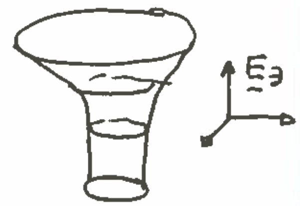
$$- mh'h''\dot{x}^2 + mA\omega\dot{x}\sin\omega t\, h''\dot{x}$$

$$= -mgh' - Kx - K(A\cos\omega t + h)h'$$

Cancelling some equal and opposite terms.

$$m(1+h'h')\ddot{x} + mh'h''\dot{x}^2 = -m(h')(g - A\omega^2\cos\omega t) - Kx - K(A\cos\omega t + h)h'$$

QUESTION 2



(a) $\quad q^1 = r, \quad q^2 = \theta \quad\quad q^3 = \eta = z - f(r)$

$$\underline{r} = r\cos\theta\,\underline{E}_1 + r\sin\theta\,\underline{E}_2 + (\eta + f)\,\underline{E}_3$$

$$\underline{v} = \dot{r}\,(\cos\theta\,\underline{E}_1 + \sin\theta\,\underline{E}_2) + \dot{r}f'\,\underline{E}_3 + r\dot{\theta}(-\sin\theta\,\underline{E}_1 + \cos\theta\,\underline{E}_2) + \dot{\eta}\,\underline{E}_3$$

Hence as $\quad \underline{v} = \sum_{\ell=1}^{3} \dot{q}^\ell \underline{a}_\ell$

$$\underline{a}_1 = \underline{e}_r + f'\underline{E}_3 \qquad \underline{a}_2 = r\underline{e}_\theta \qquad \underline{a}_3 = \underline{E}_3$$

$$[a_{ik}] = \begin{bmatrix} 1+f'f' & 0 & f' \\ 0 & r^2 & 0 \\ f' & 0 & 1 \end{bmatrix}$$

(b) $\quad \underline{a}^i = \nabla q^i \qquad$ In cylindrical polar coordinates $\nabla u = \frac{\partial u}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\underline{e}_\theta + \frac{\partial u}{\partial z}\underline{E}_3$

$$\underline{a}^1 = \nabla r = \underline{e}_r \qquad \underline{a}^2 = \nabla\theta = \frac{1}{r}\underline{e}_\theta, \quad \underline{a}^3 = \nabla\eta = \underline{E}_3 - f'\underline{e}_r$$

(c) Easiest to use approach II here

$$\tilde{T} = \frac{1}{2}m(1+f'f')\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 \qquad (\eta = 0,\ \dot{\eta} = 0)$$

$$\tilde{u} = mg\underline{E}_3\cdot\underline{r} = mg(\eta + f) = mgf$$

Hence

$$\frac{d}{dt}\left(\frac{\partial \tilde{T}}{\partial \dot{r}} = m(1+f'f')\dot{r}\right) - \left(\frac{\partial \tilde{T}}{\partial r} = mr\dot{\theta}^2 + mf'f''\dot{r}^2\right)$$
$$= -\frac{\partial \tilde{u}}{\partial r} = -mgf'$$

$$\frac{d}{dt}\left(\frac{\partial \tilde{T}}{\partial \dot{\theta}} = mr^2\dot{\theta}\right) - \left(\frac{\partial \tilde{T}}{\partial \theta} = 0\right) = -\frac{\partial \tilde{u}}{\partial \theta} = 0$$

Simplifying

$$m(1+f'f')\ddot{r} - mr\dot{\theta}^2 + mf'f''\dot{r}^2 = -mgf'$$
$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

(d) (i) $\dot{E} = \underline{F}_{nc}\cdot\underline{v} = \lambda\underline{a}^3\cdot\underline{v} = 0$ because $\underline{v} = \dot{q}^1\underline{a}_1 + \dot{q}^2\underline{a}_2$

Hence $E = \tilde{T} + \tilde{U}$ is conserved.

$$\frac{d}{dt}\left(\underline{H}_o\cdot\underline{E}_3 = mr^2\dot{\theta}\right) = \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \quad \text{from LEM.}$$

alternatively $\frac{d}{dt}\left(\underline{H}_o\cdot\underline{E}_3\right) = \dot{\underline{H}}_o\cdot\underline{E}_3 = (\underline{r}\times\underline{F})\cdot\underline{E}_3$

$$= (\underline{r}\times(\lambda\underline{a}^3 - mg\underline{E}_3))\cdot\underline{E}_3$$

$$= (\underline{r}\times\lambda\underline{a}^3)\cdot\underline{E}_3$$

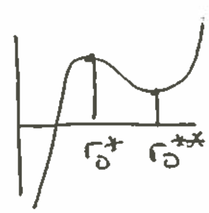
$= 0$ because $\underline{r}\times\lambda\underline{a}^3 \parallel \underline{e}_\theta$

(d) (ii) Equilibria occur when $\theta = \theta_0$, $\theta' = 0$, $r = r_0$, $r' = 0$

From EOM we see that these conditions are satisfied when

$f' = 0$ and θ_0 is arbitrary.

Hence for $f(r)$



we find 2 circles of equilibria: one for $r_0^* = r_0$ and the other for $r_0^{**} = r_0$.

There are ∞ many equilibria because θ is arbitrary.