Second Midterm Examination
Monday April 11, 2016
Closed Books and Closed Notes

Question 1 *Planar Kinematics of a System of Three Particles* *(25 Points)*

As shown in Figure 1, a system of 3 particles of equal mass are connected by identical rigid massless rods of length ℓ to a central point C. The system is free to move on a vertical plane and is subject to a gravitational force. In addition, a constant force $P\mathbf{E}_x$ acts on one of the particles as shown in the figure. To describe the kinematics of this system, the position vector of the center of mass C is described using a set of Cartesian coordinates and the position vectors of m_1, m_2, and m_3 relative to C are described using a set of cylindrical polar coordinates:

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y, \qquad \mathbf{r}_1 - \mathbf{r} = \ell\mathbf{e}_r,$$
$$\mathbf{r}_2 - \mathbf{r} = \ell\left(\cos\left(120^\circ\right)\mathbf{e}_r + \sin\left(120^\circ\right)\mathbf{e}_\theta\right), \quad \mathbf{r}_3 - \mathbf{r} = \ell\left(\cos\left(240^\circ\right)\mathbf{e}_r + \sin\left(240^\circ\right)\mathbf{e}_\theta\right). \tag{1}$$

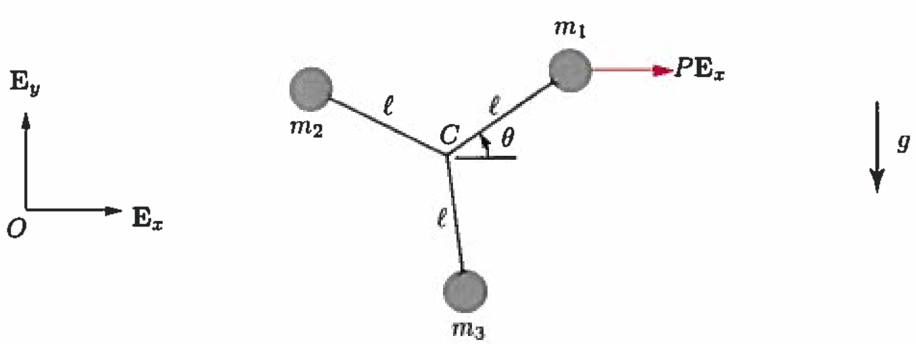


Figure 1: A system of three particles connected by massless rods of length ℓ to a common point C. The particles are free to move on a smooth vertical plane.

(a) *(5+5+5 Points)* Starting from the representations (1) and using the definitions of the linear momentum $\mathbf{G}$, angular momentum $\mathbf{H}_C$ relative to the center of mass, and kinetic energy T, show that

$$\mathbf{G} = m\left(\dot{x}\mathbf{E}_x + \dot{y}\mathbf{E}_y\right),$$
$$\mathbf{H}_C = ?,$$
$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}m??, \tag{2}$$

where the mass m of the system is $m = m_1 + m_2 + m_3$. For full credit, supply the missing terms.

(b) *(5 Points)* With the help of (2), establish a representation for the angular momentum of the system of particles relative to the fixed point O.

(c) *(5 Points)* Establish an expression for the total energy of the system.

Question 2 *Planar Dynamics of a System of Two Particles* *(25 Points)*

As shown in Figure 2, a pendulum of length ℓ and mass m_2 is free to move inside a housing of mass m_1. The housing is suspended by a spring of stiffness K and unstretched length $\ell_0 = 0$ and is free to move in the $\mathbf{E}_x$ direction with the help of frictionless guides. To describe the motion of the particles, the following representations are used:

$$\mathbf{r}_1 = x\mathbf{E}_x + \mathbf{c}, \qquad \mathbf{r}_2 = x\mathbf{E}_x + \ell\mathbf{e}_r, \qquad \mathbf{r}_A = x\mathbf{E}_x, \tag{3}$$

where $\mathbf{c}$ is a constant and A is the point of attachment of the spring to the housing. The point A is also the location of the pin joint for the pendulum.

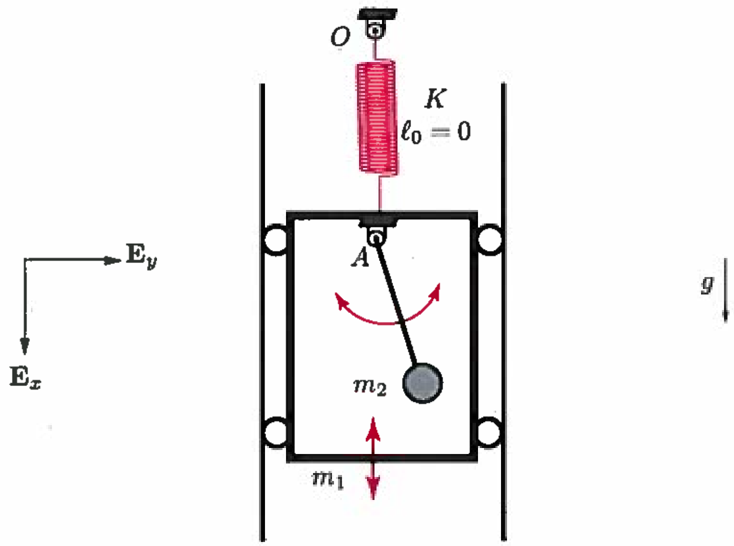


Figure 2: A housing of mass m_1 is free to move in a smooth vertical guide and is suspended from a fixed point O by a spring. A pendulum of mass m_2 is mounted inside the housing and is free to perform planar oscillations about the point A.

(a) *(5 Points)* Show that the kinetic energy of the system of particles has the representation

$$T = \frac{m_1 + m_2}{2}\dot{x}^2 + ? + ?? \tag{4}$$

For full credit supply the missing terms, some of which may be negative.

(b) *(5 Points)* Draw freebody diagrams for the individual particles. In your solution, provide a clear expression for the spring force.

(c) *(5 Points)* Show that the differential equations governing the motion of the particles are

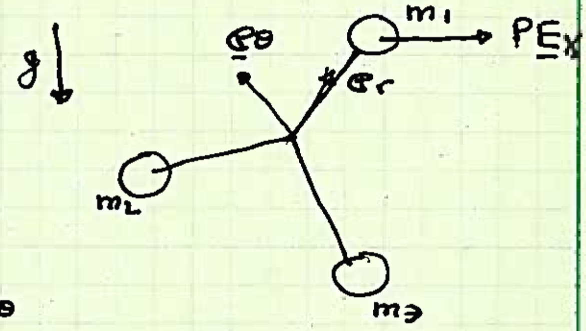
$$(m_1 + m_2)\,\ddot{x} + ??? + ???? = -Kx + (m_1 + m_2)\,g, \qquad m_2\ell\ddot{\theta} - m_2\ddot{x}????? = ?????? \tag{5}$$

For full credit, supply the missing terms, some of which may be negative.

(d) *(5 Points)* Using the work-energy theorem $\dot{T} = \mathbf{F}_1 \cdot \mathbf{v}_1 + \mathbf{F}_2 \cdot \mathbf{v}_2$, prove that the total energy of the system is conserved. For full credit, a clear expression for the total energy must be provided.

(e) *(5 Points)* Show that if the system is at rest with $\theta = 0$ and $x = \frac{(m_1+m_2)g}{K}$, then it will remain at rest. Give a physical interpretation of this rest state.

QUESTION 1



(a)

$$\underline{r}_1 - \underline{r} = \ell \underline{e}_r$$
$$\underline{v}_1 - \underline{v} = \ell \dot{\theta} \underline{e}_r$$

$$\underline{r}_2 - \underline{r} = \ell \cos 120 \underline{e}_r + \ell \sin 120^\circ \underline{e}_\theta$$
$$\underline{v}_2 - \underline{v} = \ell \dot{\theta} \cos 120 \underline{e}_\theta - \ell \dot{\theta} \sin 120^\circ \underline{e}_r$$
$$\underline{r}_2 - \underline{r} = \ell \cos 240 \underline{e}_r + \ell \sin 240 \underline{e}_\theta$$
$$\underline{v}_2 - \underline{v} = \ell \dot{\theta} \cos 240 \underline{e}_\theta - \ell \dot{\theta} \sin 240 \underline{e}_r$$

$$\underline{G} = m\dot{\underline{r}} = m\dot{x}\underline{E}_x + m\dot{y}\underline{E}_y \qquad m = m_1 + m_2 + m_3$$

$$\underline{H}_c = \sum_{i=1}^{3} (\underline{r}_i - \underline{r}) \times m_i (\underline{v}_i - \underline{v})$$
$$= m_1 \ell^2 \dot{\theta} \underline{E}_z + m_2 \ell^2 \dot{\theta} (\cos^2 120 + \sin^2 120) \underline{E}_z$$
$$+ m_3 \ell^2 \dot{\theta} (\cos^2 240 + \sin^2 240) \underline{E}_z$$
$$= m \ell^2 \dot{\theta} \underline{E}_z$$

$$T = \sum \tfrac{1}{2} m_i (\underline{v}_i - \underline{v}) \cdot (\underline{v}_i - \underline{v}) + \tfrac{1}{2} m \underline{v} \cdot \underline{v}$$
$$= \tfrac{1}{2} m_1 \ell^2 \dot{\theta}^2 + \tfrac{1}{2} m_2 \ell^2 \dot{\theta}^2 + \tfrac{1}{2} m_3 \ell^2 \dot{\theta}^2 + \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$
$$= \tfrac{1}{2} m \ell^2 \dot{\theta}^2 + \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

(b) $$\underline{H}_o = \underline{H}_c + \underline{r} \times m v = m \ell^2 \dot{\theta} \underline{E}_z + m (x\dot{y} - y\dot{x}) \underline{E}_z$$

(c) $$E = T + (m_1 g \underline{r}_1 + m_2 g \underline{r}_2 + m_3 g \underline{r}_3) \cdot \underline{E}_y - P \underline{E}_x \cdot \underline{r}_1$$
$$= T + m g \underline{r} \cdot \underline{E}_y - P \underline{r} \cdot \underline{E}_x - P \ell \cos \theta$$

+ Note that we are using the easiest representation for T and $\underline{H}_c$ to solve (a).

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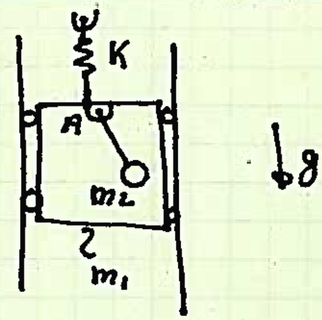
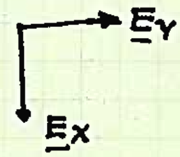
QUESTION 2





(a)

$$\underline{r}_1 = x\underline{E}_x + \underline{c}$$

$$\underline{v}_1 = \dot{x}\underline{E}_x$$

$$\underline{r}_2 = x\underline{E}_x + \ell\underline{e}_r$$

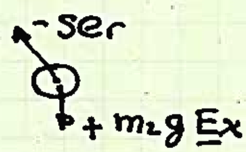
$$\underline{v}_2 = \dot{x}\underline{E}_x + \ell\dot{\theta}\underline{e}_\theta$$

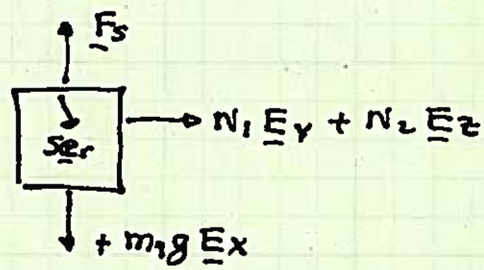
$$T = \tfrac{1}{2}m_1\underline{v}_1\cdot\underline{v}_1 + \tfrac{1}{2}m_2\underline{v}_2\cdot\underline{v}_2$$

$$= \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}m_2(\dot{x}^2 + \ell^2\dot{\theta}^2 \mp 2\ell\dot{x}\dot{\theta}\sin\theta)$$

$$= \tfrac{1}{2}(m_1+m_2)\dot{x}^2 + \tfrac{1}{2}m_2\ell^2\dot{\theta}^2 - m_2\ell\dot{x}\dot{\theta}\sin\theta$$

(b)





$$\underline{F}_s = -K(\|\underline{r}_A\| - \ell_0)\frac{\underline{r}_A}{\|\underline{r}_A\|} = -K\underline{r}_A = -Kx\underline{E}_x$$

(c) $\underline{F} = m\underline{a}$ system:

$$\underline{F}_s + (m_1 g + m_2 g)\underline{E}_x + N_1\underline{E}_y + N_2\underline{E}_z$$

$$= m_1\ddot{\underline{r}}_1 + m_2\ddot{\underline{r}}_2$$

$$= (m_1+m_2)\ddot{x}\underline{E}_x + m_2\ell\ddot{\theta}\underline{e}_\theta - m_2\ell\dot{\theta}^2\underline{e}_r$$

$\cdot\underline{E}_x$:

$$\boxed{-Kx + (m_1+m_2)g = (m_1+m_2)\ddot{x} \mp m_2\ell\ddot{\theta}\sin\theta - m_2\ell\dot{\theta}^2\cos\theta}$$

This is $(5)_1 \equiv (\underline{F} = m\underline{a})\cdot\underline{E}_x$ for system.

$\underline{F}_2 = m_2 \underline{a}_2$:

$$-S\underline{e}_r + m_2 g \underline{E}_x = m_2 \ddot{x} \underline{E}_x + m_2 \ell \ddot{\theta} \underline{e}_\theta - m_2 \ell \dot{\theta}^2 \underline{e}_r$$

$\cdot \, \underline{e}_\theta$

$$\boxed{-m_2 g \sin\theta = -m_2 \ddot{x} \sin\theta + m_2 \ell \ddot{\theta}}$$

This is $(5)_2 \equiv (\underline{F}_2 = m_2 \underline{a}_2) \cdot \underline{e}_\theta$ for the pendulum.

(d) $$\dot{T} = \underline{F}_1 \cdot \underline{v}_1 + \underline{F}_2 \cdot \underline{v}_2$$

$$= \underline{F}_s \cdot \underline{v}_1 + m_1 g \underline{E}_x \cdot \underline{v}_1 + (N_1 \underline{E}_y + N_2 \underline{E}_z) \cdot \underline{v}_1$$

$$+ S\underline{e}_r \cdot \underline{v}_1 - S\underline{e}_r \cdot \underline{v}_2 + m_2 g \underline{E}_x \cdot \underline{v}_2$$

$$= -\frac{d}{dt}\left(\tfrac{1}{2} K x^2 \mp (m_1 g \underline{r}_1 + m_2 g \underline{r}_2) \cdot \underline{E}_x \right) + 0$$

$$+ S\underline{e}_r \cdot (\underline{v}_1 - \underline{v}_2) \longrightarrow$$ Goes to zero because $S\underline{e}_r \perp \underline{v}_1 - \underline{v}_2$

$$= -\frac{d}{dt}(u) + 0$$

$$\Rightarrow \frac{d}{dt}\left(E = T + u = T + \tfrac{1}{2} K x^2 \mp (m_1 g \underline{r}_1 + m_2 g \underline{r}_2) \cdot \underline{E}_x \right) = 0$$

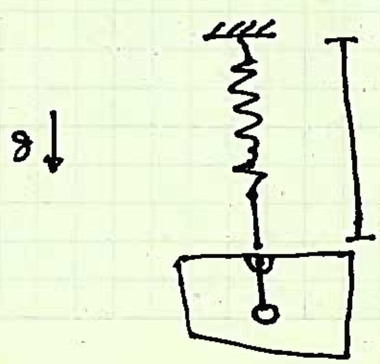
(e) From Eon of motion from (E):

$$0 = (m_1 + m_2)\ddot{x}$$

$$0 = 0 + m_2 \ell \ddot{\theta}$$

$\Rightarrow$ $\ddot{\theta}$ and $\ddot{x}$ are zero so $\dot{\theta}$ and $\dot{x}$ remain zero and so θ and x remain at their equilibrium values.

The system is at rest with the spring extended to balance gravity



$$x = \frac{m_1 g + m_2 g}{K}$$