First Midterm Examination
Closed Books and Closed Notes
Answer Both Questions for Maximum Credit

Question 1
A Particle in Motion on a Horizontal Plane (25 POINTS)

As shown in Figure 1, a particle of mass m is free to move on a smooth circular track of radius R_0. The particle is connected to a fixed point A by a spring of stiffness K and unstretched length $\ell_0 = 0$. A gravitational force $-mg\mathbf{E}_z$ acts on the particle.

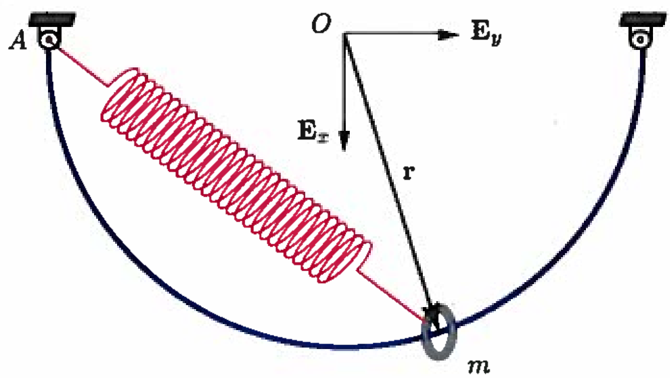


Figure 1: *Schematic of a particle of mass m which is free to move on a circular track of radius R_0. A vertical gravitational force $-mg\mathbf{E}_z$ acts on the particle.*

(a) Starting from the following representation for the position vector $\mathbf{r}$,

$$\mathbf{r} = R_0\mathbf{e}_r + 0\mathbf{E}_z, \tag{1}$$

establish expressions for the velocity vector $\mathbf{v}$ and acceleration vector $\mathbf{a}$ of the particle.

(b) Draw a freebody diagram of the particle in motion. Your freebody diagram should include a clear expression for the spring force.

(c) Suppose that the particle is in motion on the track. Show that the differential equations governing the motion of the particle and the normal force acting on the particle are

$$mR_0\ddot{\theta} = -KR_0????, \qquad \mathbf{N} = (?+??)\,\mathbf{e}_r + ??\mathbf{E}_z. \tag{2}$$

For full credit, supply the missing terms.

(d) Suppose that the particle is released from rest when $\theta = 0$. Show that the particle will reach A with a finite speed that is proportional to $\sqrt{\frac{K}{m}}$.

Question 2

A Particle Moving on Plane Curve (25 POINTS)

As shown in Figure 2, a particle of mass m is in motion on a rough plane curve:

$$\mathbf{r} = x\mathbf{E}_x + \frac{\alpha}{2}x^2\mathbf{E}_y, \tag{3}$$

where α is a constant. A vertical gravitational force $-mg\mathbf{E}_y$ acts on the particle. The curvature and unit normal vector to the path have the representations:

$$\kappa = \frac{|\alpha|}{\left(\sqrt{1+\alpha^2x^2}\right)^3}, \qquad \mathbf{e}_n = \frac{\text{sgn}(\alpha)}{\sqrt{1+\alpha^2x^2}}\left(-\alpha x\mathbf{E}_x + \mathbf{E}_y\right). \tag{4}$$

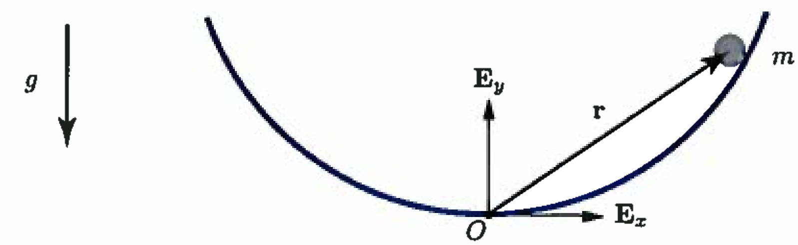


Figure 2: *Schematic of a particle of mass m which is free to move on the upper side of a rough curve $y = 0.5\alpha x^2$. The case $\alpha > 0$ is shown.*

(a) Suppose the particle is in motion on the curve with $\dot{x} > 0$. Starting from (3) establish expressions for the velocity $\mathbf{v}$ and acceleration vector $\mathbf{a}$ for the particle. Show that

$$v = \dot{x}\sqrt{1+\alpha^2x^2}, \qquad \mathbf{e}_t = \frac{1}{\sqrt{1+\alpha^2x^2}}\left(\mathbf{E}_x + \alpha x\mathbf{E}_y\right). \tag{5}$$

(b) Assuming that the particle is moving on the rough curve, draw a freebody diagram of the particle. Give a clear expression for the friction force and assume that $\dot{x} > 0$.

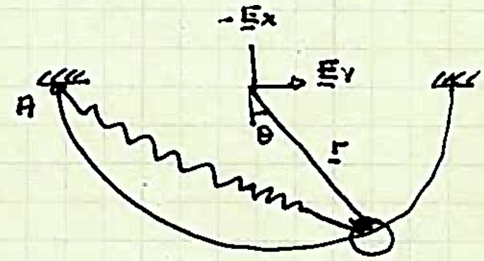
(c) Show that the differential equation governing the motion of the particle and the normal force acting on the particle are given by the following expressions:

$$m\sqrt{1+\alpha^2x^2}\ddot{x} = m\dot{x}^2? - mg\,(??) - \mu_k??? \qquad \mathbf{N} = m\,(g????+?????)\,\mathbf{e}_n. \tag{6}$$

For full credit supply the missing terms. In addition, establish a criterion for the particle to remain on the curve for the cases $\text{sgn}(\alpha) = 1$ and $\text{sgn}(\alpha) = -1$

(d) Suppose that the particle is instantaneously at rest. Show how a criterion featuring μ_s and α can be established which, if satisfied, indicates that the particle will remain at rest. Illustrate your criterion with a sketch for the case $\alpha > 0$.

QUESTION 1



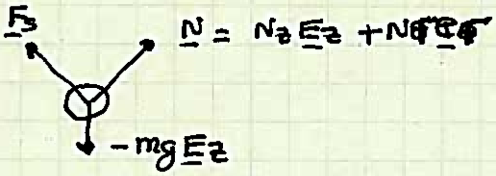
(a)

$$\underline{r} = R_0 \underline{e}_r$$

$$\underline{v} = R_0 \dot{\theta} \underline{e}_\theta$$

$$\underline{a} = R_0 \ddot{\theta} \underline{e}_\theta - R_0 \dot{\theta}^2 \underline{e}_r$$

(b)



$$\underline{F}_s = -K\left(\|\underline{r}-\underline{r}_A\|\right)\frac{\underline{r}-\underline{r}_A}{\|\underline{r}-\underline{r}_A\|} = -K\left(\underline{r}-\underline{r}_A\right)$$

$$= -K\left(R_0 \underline{e}_r + R_0\left(\cos\theta\, \underline{e}_\theta + \sin\theta\, \underline{e}_r = \underline{E}_y\right)\right)$$

$$= -K R_0\left(\underline{e}_r + \underline{E}_y\right)$$

(c) $\underline{F} = m\underline{a}$

$\cdot\, \underline{e}_\theta$ $\quad m R_0 \ddot{\theta} = -K R_0 \underline{E}_y \cdot \underline{e}_\theta = -K R_0 \cos\theta$

$\cdot\, \underline{e}_r$ $\quad N_r + \underline{F}_s \cdot \underline{e}_r = -m R_0 \dot{\theta}^2$

$\cdot\, \underline{E}_z$ $\quad N_z + (-mg) = 0$

Hence

$$m R_0 \ddot{\theta} = -K R_0 \cos\theta$$

$$\underline{N} = mg\underline{E}_z + K R_0(1+\sin\theta)\underline{e}_r - m R_0 \dot{\theta}^2 \underline{e}_r$$

(d) From $\ddot{\theta} = -\frac{K}{m}\cos\theta$ we use identity $a\,ds = v\,dv$

to find

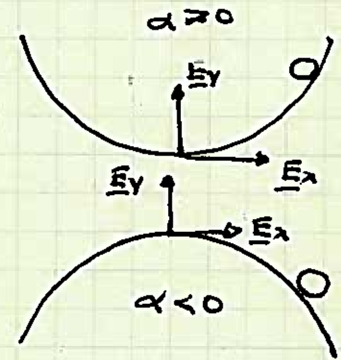
$$\frac{1}{2}\dot{\theta}_A^2 - \frac{1}{2}\dot{\theta}_0^2 = -\frac{K}{m}\sin\theta_A + \frac{K}{m}\sin\theta_0$$

If particle is released from rest when $\theta_0 = 0$, $\dot{\theta}_0 = 0$, then when $\theta_A = -\pi/2$

$$\dot{\theta}_A^2 = \frac{2K}{m}$$

Hence $\dot{\theta}_A = -\sqrt{\frac{2K}{m}}$. and $\underline{v}_A = -R_0\sqrt{\frac{2K}{m}}\,\underline{E}_x$

QUESTION 2



(a) $\underline{r} = x\,\underline{E}_x + \frac{\alpha}{2} x^2 \underline{E}_y$

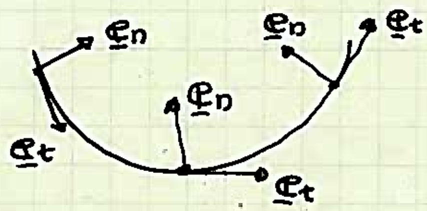
$\underline{v} = \dot{x}\,\underline{E}_x + \frac{2}{2}\alpha x \dot{x}\,\underline{E}_y$

$\underline{a} = \ddot{x}\,\underline{E}_x + \alpha x \ddot{x}\,\underline{E}_y + \alpha \dot{x}^2 \underline{E}_y$

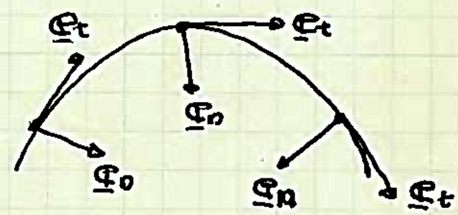
(b) Hence $v = \|\underline{v}\| = \dot{x}\sqrt{1 + x^2\alpha^2}$

$\underline{e}_t = \frac{\underline{v}}{v} = \frac{1}{\sqrt{1+\alpha^2 x^2}}\left(\underline{E}_x + \alpha x\,\underline{E}_y \right)$

(c)

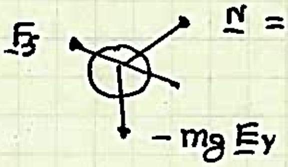


$\alpha = 1$



$\alpha = -1$

(b)



$\underline{N} = N_n \underline{e}_n + N_b \underline{e}_b$

$\underline{F}_f = -\mu_K \|\underline{N}\|\,\underline{e}_t$

(d) $\underline{F} = m\underline{a}$

$\cdot\,\underline{e}_t \qquad m\dot{v} = -\mu_K \|\underline{N}\| - mg\,\underline{E}_y \cdot \underline{e}_t$

$= -\mu_K \|\underline{N}\| - \frac{mg\,\alpha x}{\sqrt{1+\alpha^2 x^2}}$

$\cdot\,\underline{e}_n \qquad N_n = \left(mg\,\underline{E}_y \cdot \underline{e}_n = \frac{mg\,\mathrm{sgn}(\alpha)}{\sqrt{1+\alpha^2 x^2}} \right) + m\kappa v^2$

$\cdot\,\underline{e}_b \qquad N_b = 0$

TOPS

Now

$$\dot{v} = \ddot{x}\sqrt{1+\alpha^2x^2} + \frac{x\dot{x}^2\alpha^2}{\sqrt{1+x^2\alpha^2}}$$

[Computing $\dot{v}$ by differentiating v]

$$= \underline{a}\cdot\underline{e}_t = \left(\ddot{x}\,\underline{E}_x + \alpha(\dot{x}^2 + x\ddot{x})\underline{E}_y\right)\cdot\underline{e}_t$$

[alternative method of computing $\dot{v}$]

Hence

$$m\sqrt{1+\alpha^2x^2}\,\ddot{x} = -\frac{mx\alpha^2\dot{x}^2}{\sqrt{1+\alpha^2x^2}} - \frac{mg\alpha x}{\sqrt{1+\alpha^2x^2}} - \mu_K\|\underline{N}\|$$

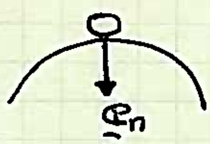
where $$\|\underline{N}\| = \left|\frac{mg\,\mathrm{sign}\,\alpha}{\sqrt{1+\alpha^2x^2}} + mKv^2\right|$$

For particle to remain on curve

when $\alpha > 0$ we need $N_n > 0$ $\Rightarrow$ $mKv^2 + \dfrac{mg}{\sqrt{1+\alpha^2x^2}} > 0$

which is always true (as expected)

when $\alpha < 0$ we need $N_n < 0$ $\quad \dfrac{mg}{\sqrt{1+\alpha^2x^2}} - mKv^2 \geq 0$

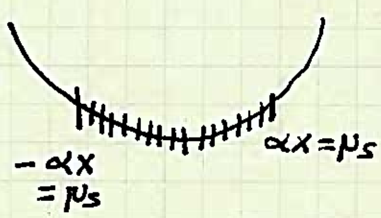


(d) Friction is static $\underline{a} = \underline{0}$ $\quad F_f\underline{e}_t + \underline{N} - mg\underline{E}_y = \underline{0}$

$\Rightarrow$ $F_f = \dfrac{mg\alpha x}{\sqrt{1+\alpha^2x^2}}$ $\quad \underline{N} = \dfrac{mg\,\mathrm{sign}(\alpha)}{\sqrt{1+\alpha^2x^2}}\underline{e}_n$

Static friction criterion : $|F_f| \leq \mu_s\|\underline{N}\|$

$\Rightarrow$ $|\alpha x| \leq \mu_s$



Sticking possible in hatched area.

Otherwise if particle is placed further up incline there is insufficient static friction for it to remain at rest there.