University of California, Berkeley
Mechanical Engineering
E117 Engineering analysis **SOLUTIONS**
1st Test F15 *Prof S. Morris*

1.(65) Find the general **real** solution of the equation $u'''(x) + u(x) = 0$. The dependent variable x is real, $-\infty < x < \infty$.

Solution

Step 1. Basis of solutions.

Because the equation is linear and has constant coefficients, it admits the solution $u = \mathbf{e}^{px}$ (constant p to be determined). By substitution, we find that $\mathbf{e}^{px}$ satisfies the differential equation if

$$p^3 = -1. \tag{1.1}$$

Because (1.1) has 3 distinct roots, the differential equation has 3 linearly independent exponential solutions.

To solve (1.1), let $p = r\mathbf{e}^{\mathrm{i}\phi}$, r real and non–negative. Then

$$\begin{aligned} p^3 =& r^3\mathbf{e}^{3\mathrm{i}\phi} \qquad \text{(by Lect.3, Eq.3)} \\ =& r^3\{\cos 3\phi + \mathrm{i}\sin 3\phi\} \qquad \text{(Euler formula, Lect.4, Eq.12.)} \end{aligned} \tag{1.2}$$

Substituting (1.2) into (1.1), we obtain

$$r^3\mathbf{e}^{3\mathrm{i}\phi} = -1. \tag{1.3}$$

Taking the magnitude of both sides, and using $|\mathbf{e}^{\mathrm{i}\phi}| = 1$ (ϕ real), we find that $r = 1$. Consequently,

$$\mathbf{e}^{3\mathrm{i}\phi} = -1, \ \Rightarrow \ \cos 3\phi + \mathrm{i}\sin 3\phi = -1. \tag{1.4a,b}$$

Equating real and imaginary parts, we obtain

$$\cos 3\phi = -1 \ \ \Rightarrow 3\phi = \pi, 3\pi, 5\pi, \dots \tag{1.5a}$$

$$\sin 3\phi = 0 \tag{1.5b}$$

Values given by (1.5a) also satisfy (1.5b); so $\phi = \pi/3, \pi, 5\pi/3$, all further values ($7\pi/3, \dots$ are equivalent to one of those three).

Using $\cos\frac{\pi}{3} = \frac{1}{2}$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, we obtain 3 roots:

$$p_1 = -1, \tag{1.6a}$$

$$p_{2,3} = \mathbf{e}^{\pm\mathrm{i}\pi/3}, = \tfrac{1}{2} \pm \mathrm{i}\tfrac{\sqrt{3}}{2}; \tag{1.6b,c}$$

These roots lie on the unit circle in the complex plane; they trisect the unit circle. Because Eq.(1.1) has real coefficients, roots $p_{2,3}$ are complex conjugates.

Step 2. Real solution.

Method 1. Let c_0, c_1 be, respectively, a real constant and a complex constant. Then the general real solution of the differential equation is

$$u(x) = c_0\mathbf{e}^{-x} + \mathbf{e}^{x/2}\{c_1\mathbf{e}^{\mathrm{i}x\sqrt{3}/2} + \bar{c}_1\mathbf{e}^{-\mathrm{i}x\sqrt{3}/2}\}. \tag{1.7a}$$

(As in the class notes, Lect.3, $\bar{c}_1$ denotes the complex conjugate of c_1.)

Method 2. Form a basis of real solutions; then superpose using real constants c_0, c_2, c_3. From the functions $\mathbf{e}^{x/2}\mathbf{e}^{\pm\mathrm{i}x\sqrt{3}/2}$, form the linearly independent real functions $\mathbf{e}^{x/2}\cos x\frac{\sqrt{3}}{2}$, $e^{x/2}\sin x\frac{\sqrt{3}}{2}$. The general real solution is

$$u(x) = c_0\mathbf{e}^{-x} + \mathbf{e}^{x/2}\{c_2\cos x\frac{\sqrt{3}}{2} + c_3\sin x\frac{\sqrt{3}}{2}\}. \tag{1.7b}$$

Equations (1.7a) and (1.7b) are equivalent: $c_2 = 2\,\mathrm{Re}\,c_1$, and $c_3 = -2\,\mathrm{Im}\,c_1$.

2.(65) For $0 < x < 1$, $f(x) = 1$. That function is extended from this interval into the domain $-\infty < x < \infty$ according to two different rules. For each case, sketch the extended function. Then calculate its Fourier series.

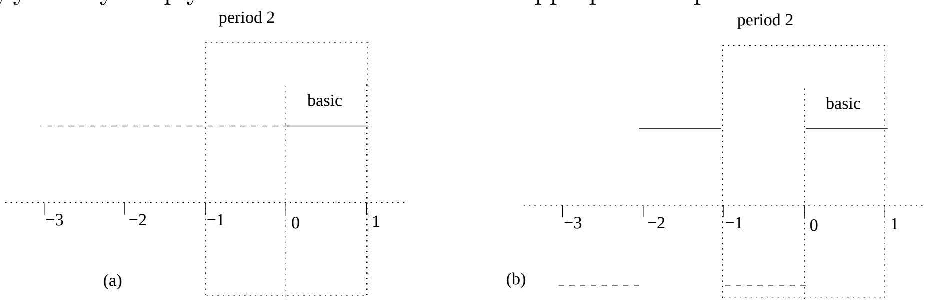
(a) The extended function has period 2, and is an even function of x.

(b) The extended function has period 2, and is an odd function of x.

Given. The Fourier series of a function $G(x)$ with period $2p$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\};\ \ a_n = \frac{1}{p}\int_d^{d+2p} G(x)\cos\frac{n\pi x}{p}\,\mathrm{d}x,\ \ b_n = \frac{1}{p}\int_d^{d+2p} G(x)\sin\frac{n\pi x}{p}\,\mathrm{d}x.$$

Note. If, in either case (a) or (b), you can see the correct answer from your sketch, you need not provide a calculation; you may simply state the answer with the appropriate explanation.



Solution nn Figs.(a) and (b) show, respectively, the even extension and the odd extension.

For case (a), the extended function is equal to unity everywhere; because its average over a period is equal to unity, $a_0 = 2$ and all other Fourier coefficients vanish. The Fourier cosine half–range expansion of $f(x) = 1$ $(0 < x < 1)$ is, therefore

$$1 = 1. \tag{2.1}$$

For case (b),the extended function is a square wave. Because the extended function is odd with period 2, the coefficients $a_n = 0$. To calculate b_n, we set the period $2p = 2$ so $p = 1$ and choose $d = -1$. So

$$\begin{aligned} b_n &= \int_{-1}^{1} f_{odd}(x)\sin n\pi x\,\mathrm{d}x, \\ &=2\int_0^1 \sin n\pi x\,\mathrm{d}x \\ &=\frac{2}{n\pi}[-\cos n\pi x]_0^1 \end{aligned}$$

So

$$b_n = \begin{cases} 0 & \text{if } n = 2k \\ \frac{4}{\pi(2k+1)} & \text{if } n = 2k+1 \end{cases}$$

The Fourier sine half–range expansion of $f(x) = 1$ $(0 < x < 1)$ is, therefore,

$$1 = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{2k+1}\sin(2k+1)\pi x. \tag{2.2}$$

3.(70) For $0 < x < 1$ and $0 < t < \infty$, $T(x,t)$ satisfies

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}. \tag{3.1a}$$

$$\text{At } x = 0 \text{ and at } x = 1, \ \ T = 0. \tag{3.1b,c}$$

$$\text{At } t = 0, \text{ and for } 0 < x < 1, \ \ T = \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin(2k+1)\pi x. \tag{3.1d}$$

Using separation of variables and superposition, find $T(x,t)$.

Note. If, based on physical reasoning, you make assumptions about the separation constant, you must explain briefly your reasoning. But you are **not** asked to prove analytically that the separation constant is real; do not waste time.

Solution

Step 1. Basis of solutions. Substituting $T(x,t) = A(t)X(x)$ into (3.1a), then separating variables, we obtain

$$\frac{\dot{A}}{A} = \frac{X''}{X}. \tag{3.2}$$

Because the left side of (3.2) contains only functions of t, but the right hand side contains only functions of x, each side must be constant.

Because we expect heat conduction to eliminate temperature differences, we denote the separation constant by $-\lambda$, and expect λ to be a positive real number. (As stressed in class, λ is determined by the solution of the following eigenvalue problem; representing the constant in the form $-\lambda$ is a notational convenience.)

With that choice of notation, λ and X are determined by the solution of the eigenvalue problem: for $0 < x < 1$,

$$X''(x) + \lambda X(x) = 0. \tag{3.3a}$$

$$X(0) = 0 = X(1). \tag{3.3b,c}$$

The general solution of (3.3a) is

$$X = c_0 \cos x\sqrt{\lambda} + c_1 \sin x\sqrt{\lambda}$$

Because $\sin 0 = 0, \cos 0 = 1$, boundary condition (3.3b) requires $c_0 = 1$. Because $c_1 \neq 0$, boundary condition (3.3c) requires

$$\sin\sqrt{\lambda} = 0. \tag{3.4}$$

From Lect.4-12, the only zeros of $\sin z$ lie along the real axis: so $\sqrt{\lambda} = n\pi$, $n = 1, 2, 3\ldots$. The solution of the eigenvalue problem (3.3) is, therefore,

$$X = \sin n\pi x, \ \ \lambda = n^2\pi^2. \tag{3.5}$$

A now satisfies the equation $\dot{A} + n^2\pi^2 A = 0$, so $A = \mathbf{e}^{-n^2\pi^2 t}$. The basis of separable solutions of (3.1a) and boundary conditions (3.1b,c) is, therefore,

$$T_n(x,t) = \mathbf{e}^{-n^2\pi^2 t}\sin n\pi x. \tag{3.6}$$

Step 2. Fitting initial conditions (3.1d).

By the principle of superposition, the general expression satisfying (3.1a), (3.1b) and (3.1c) is

$$T(x,t) = \sum_{n=1}^{\infty} b_n \mathbf{e}^{-n^2\pi^2 t} \sin n\pi x. \tag{3.7}$$

Setting $t = 0$ in (3.7), and equating to the initial condition (3.1d), we obtain

$$\sum_{n=1}^{\infty} b_n \sin n\pi x = T = \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin(2k+1)\pi x. \tag{3.8}$$

On the *right* hand side of (3.8), the $k = 0$ term is proportional to $\sin \pi x$; therefore, that series defines the odd periodic extension of the initial conditions holding on $0 < x < 1$. On the *left* side of (3.8), $n = 1$ term is also proportional to $\sin \pi x$; so, that series also defines the odd periodic extension of the initial conditions. The coefficients in the Fourier series can, therefore, be equated:

$$b_n = \begin{cases} 0 & \text{for } n = 2k, \\ \frac{4}{\pi^2} \frac{(-1)^k}{(2k+1)^2} & \text{for } n = 2k+1 \end{cases} \tag{3.9}$$

Substituting (3.9) into (3.7), we obtain the solution:

$$T(x,t) = \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \mathbf{e}^{-(2k+1)^2\pi^2 t} \sin(2k+1)\pi x. \tag{3.10}$$

Note. Not part of the test: the Fourier series given for the initial condition corresponds to

$$T(x,0) = \begin{cases} x & \text{for } 0 < x < \frac{1}{2}, \\ 1 - x & \text{for } \frac{1}{2} < x < 1. \end{cases}$$

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