

## Problem 1

(a) 5pts

$$E_{int} = \frac{d}{2} N k_B T$$

$$C_V = \frac{d}{2} R$$

where  $d$  is the number of degrees of freedom. In this problem,  $d = 5$ .

Grading Rubric:

- +3 Correct expression for  $E_{int}$
- (+1 Including  $\frac{d}{2}$  term)
- (+1 Showing  $E_{int} \propto T$ )
- (+1 Including  $Nk_B$ )
- (+0.5 Using  $nR$ , not  $Nk_B$ )
- +2 Correct expression for  $C_V$
- (+1 Including  $\frac{d}{2}$  term)
- (+1 Including R)

(b) 7pts

For isovolumetric process,  $dE_{int} = TdS - PdV = TdS$  ( $dV = 0$ ).

From this, we get the expression for the infinitesimal entropy change:

$$dS = \frac{dE_{int}}{T} = \frac{\frac{d}{2} N k_B dT}{T}$$

Integrating both sides,

$$\Delta S = \int dS = \int_{T_i}^{T_f} \frac{\frac{d}{2} N k_B dT}{T} = \frac{d}{2} N k_B \int_{T_i}^{T_f} \frac{dT}{T} = \frac{d}{2} N k_B \ln(T) \Big|_{T_i}^{T_f} = \frac{d}{2} N k_B \ln\left(\frac{T_f}{T_i}\right)$$

Again,  $d = 5$ .

Grading Rubric:

- +1 Showing  $dE_{int} = TdS$  ( $dV = 0$ )
- +1 Correct Expression for  $dE_{int}$
- +1 Integrating  $dS$  to get  $\Delta S$
- +2 Showing  $\Delta S = \int_{T_i}^{T_f} \frac{\frac{d}{2} N k_B dT}{T}$
- +2 Correct Answer
- 0.5 Using  $nR$ , not  $Nk_B$

(c) 8pts

For adiabatic process,  $dE_{int} = dQ - PdV = -PdV$  ( $dQ = 0$ ).

$dE_{int} = \frac{d}{2} N k_B dT$ , and from the ideal gas law,  $P = \frac{N k_B T}{V}$ . Then,

$$dE_{int} = -PdV$$

$$\frac{d}{2} N k_B dT = -\frac{N k_B T}{V} dV$$

Dividing both sides by  $Nk_B T$ ,

$$\frac{d}{2} \frac{dT}{T} = - \frac{dV}{V}$$

Integrating both sides,

$$\frac{d}{2} \int_{T_i}^{T_f} \frac{dT}{T} = - \int_{V_i}^{V_f} \frac{dV}{V}$$

$$\frac{d}{2} \ln\left(\frac{T_f}{T_i}\right) = -\ln\left(\frac{V_f}{V_i}\right)$$

$$\ln\left(\frac{T_f}{T_i}\right)^{\frac{d}{2}} = \ln\left(\frac{V_f}{V_i}\right)^{-1}$$

$$\left(\frac{T_f}{T_i}\right)^{\frac{d}{2}} = \left(\frac{V_i}{V_f}\right)$$

From the ideal gas law,  $T = \frac{PV}{Nk_B}$

$$\left(\frac{P_f V_f}{Nk_B} \cdot \frac{Nk_B}{P_f V_f}\right)^{\frac{d}{2}} = \frac{V_i}{V_f}$$

$$\left(\frac{P_f}{P_i}\right)^{\frac{d}{2}} \left(\frac{V_f}{V_i}\right)^{\frac{d}{2}} = \frac{V_i}{V_f}$$

$$\left(\frac{P_f}{P_i}\right)^{\frac{d}{2}} = \left(\frac{V_i}{V_f}\right)^{1+\frac{d}{2}}$$

$$\frac{P_f}{P_i} = \left(\frac{V_i}{V_f}\right)^{(1+\frac{d}{2}) \cdot \frac{2}{d}}$$

$$\frac{P_f}{P_i} = \left(\frac{V_i}{V_f}\right)^{\frac{2}{d}+1}$$

$$P_i V_i^{1+\frac{2}{d}} = P_f V_f^{1+\frac{2}{d}}$$

Hence, we conclude:

$$P_i V_i^\gamma = P_f V_f^\gamma$$

where  $\gamma = 1 + \frac{2}{d}$ . In this problem,  $d = 5$ , so  $\gamma = 1 + \frac{2}{5} = \frac{7}{5}$

#### Grading Rubric:

- +1 Explaining  $Q = 0$  for adiabatic process
- +1 Showing  $\frac{d}{2} Nk_B dT = -PdV$
- +2 Using the ideal gas law to write P in terms of T and V (or alternatively write dT in terms of dP and dV)
- +1 Separated the variables by moving all the T (or P) terms to one side and all the V terms to the other side and then integrated
- +2 Found a relation between P (or  $\frac{P_f}{P_i}$ ) and V (or  $\frac{V_f}{V_i}$ ), after doing the integration
- (-1 Algebra mistake)
- +1 Correct expression for  $\gamma$
- (+0.5 if student either states the relation  $\gamma = 1 + \frac{2}{d}$  without derivation or just writes  $\gamma = \frac{C_p}{C_v}$ )

## Problem 2

September 28, 2016

In this problem, you can get 20 points in total. I split it up into 10 parts that are worth 2 points each, to give you many opportunities for partial credit. You can't get partial credit for partial credit, unless you made a really tiny mistake, in which case you get 1.5 instead of 2 points for that part. An answer that doesn't have the right units is not such a tiny mistake!

Since it's mentioned explicitly in the problem text, **you only get credit if your answer is written in terms of the variables given in the problem**, and if you include an explanation for your answer.

- (a) The  $p - V$  diagram can be seen in figure 1. Note that it contains arrows, which are very important. For this diagram, you can get 2 points.

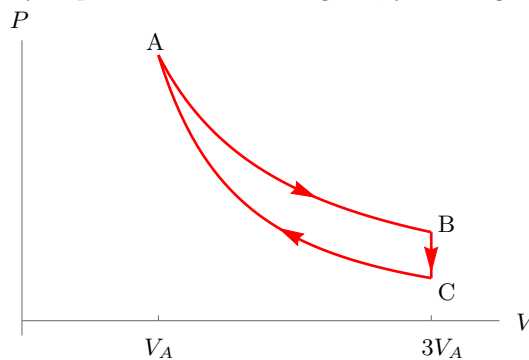


Figure 1: The physical setup of the problem

- (b) We get the following works, each is worth 2 points:

$$W_{AB} = \int p dV = nrT_A \int_{V_A}^{3V_A} \frac{dV}{V} = nRT_A \ln 3 = P_A V_A \ln 3,$$

$$W_{BC} = 0, \text{ because the volume is constant,}$$

$$\begin{aligned} W_{CA} &= -\Delta E_{\text{int}} = -\frac{3}{2}nR\Delta T = -\frac{3}{2}nR(T_A - T_C) = -\frac{3}{2}nR \left( T_A - T_A \left( \frac{V_A}{3V_A} \right)^{\gamma-1} \right) = \\ &= \frac{3}{2}nRT_A \left( 3^{-2/3} - 1 \right) = \frac{3}{2}P_A V_A \left( 3^{-2/3} - 1 \right), \end{aligned}$$

where we used that  $TV^{\gamma-1} = \text{const.}$  for the adiabatic process.

(c) We get the following heats, each is worth 2 points:

$$Q_{AB} = W_{AB} = P_A V_A \ln 3,$$

$$Q_{BC} = n c_V \Delta T = n R \frac{3}{2} (T_C - T_B) = n R \frac{3}{2} T_A (3^{-2/3} - 1) = \frac{3}{2} P_A V_A (3^{-2/3} - 1)$$

$$Q_{CA} = 0.$$

(d) We get the following entropy changes, each is worth 2 points:

$$\Delta S_{AB} = \int \frac{dQ}{T} = \frac{1}{T_A} \int dQ = \frac{Q_{AB}}{T_A} = \frac{P_A V_A}{T_A} \ln 3,$$

$$\Delta S_{BC} = \int \frac{dQ}{T} = \int_{T_B}^{T_C} \frac{n c_V dT}{T} = \frac{3}{2} n R \ln \left( \frac{T_C}{T_B} \right) = \frac{3}{2} n R \ln (3^{-2/3}) = -n R \ln 3 =$$

$$= -\frac{P_A V_A}{T_A} \ln 3$$

$$\Delta S_{CA} = 0.$$

This makes sense, since entropy is a state function, so  $\Delta S_{\text{total}} = \Delta S_{AB} + \Delta S_{BC} + \Delta S_{CA} = 0$ .

### Problem 3. A Carnot Cycle

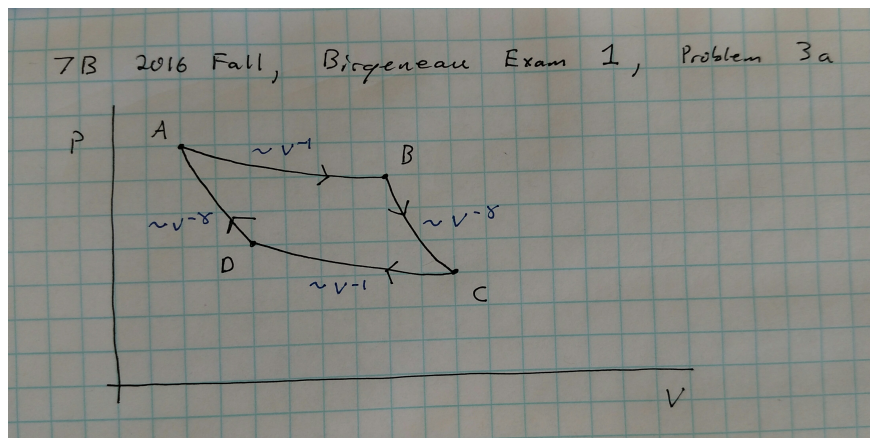
(3a) [5 points]

This cycle consists of isothermal and adiabatic paths. On the isothermal paths pressure relates to volume as  $P \sim V^{-1}$ , and on the adiabatic paths  $P \sim V^{-\gamma}$  where  $\gamma$  is the adiabatic index of the gas. Since  $\gamma > 1$ , the adiabatic curves in the  $(P, V)$  plane rise and fall more steeply than the isothermal ones.

There are four legs to consider:

- $A \rightarrow B$  is isothermal expansion, so  $V$  increases and  $P$  decreases slightly as  $P \sim V^{-1}$
- $B \rightarrow C$  is adiabatic expansion, so  $V$  increases further and  $P$  decreases significantly as  $P \sim V^{-\gamma}$
- $C \rightarrow D$  is isothermal compression, so  $V$  now decreases and  $P$  increases slightly as  $P \sim V^{-1}$
- $D \rightarrow A$  is adiabatic compression, so  $V$  again decreases back to the original volume and  $P$  increases significantly as  $P \sim V^{-\gamma}$  to the original pressure.

This gives the cycle below in the  $(P, V)$  plane.



(3b) [15 points]

The efficiency is defined as the net work done by the gas per unit input heat, so we need to calculate both the net work done  $W$  and the input heat  $Q_H$ . There are multiple way to do this, and I'll show several below.

The most direct way to compute the net work is to integrate  $dW = P dV$  over the entire cycle. In doing this we must account carefully for each leg of the cycle, as nonzero work is done during all four legs. Slightly simpler is to recall that due to the First Law (Energy Conservation) the net work done is equal to the net heat absorbed  $Q_H - Q_L$ , so we can calculate  $W$  indirectly by finding the input heat  $Q_H$  and output heat  $Q_L$ :

$$e = \frac{W}{Q_H} = \frac{Q_H - Q_L}{Q_H} = 1 - \frac{Q_L}{Q_H} \quad (1)$$

This is a bit simpler – since we know that no heat is exchanged during the adiabatic legs  $BC$  and  $DA$ , we can focus only on the isothermal legs  $AB$  and  $CD$ .

We now need the input heat  $Q_H$  absorbed over AB and the output heat  $Q_L$  exhausted during CD. Again there are several ways to compute these. A slick way is to use a trick: we know that the change in entropy of the gas is related to its heat absorbed via  $ds = dq/T$ , but also that for a closed cycle the change in entropy (zero as entropy is a state function). This gives a relation between the exchanged heats and temperatures over the cycle:

$$\begin{aligned}
 0 = \Delta S &= \int_{ABCD} \frac{1}{T} dq \\
 &= \int_{AB} \frac{1}{T} dq + \int_{CD} \frac{1}{T} dq \\
 &= \frac{1}{T_H} \int_{AB} dq + \frac{1}{T_L} \int_{CD} dq \\
 &= \frac{Q_H}{T_H} - \frac{Q_L}{T_L} \\
 \Rightarrow \frac{Q_L}{Q_H} &= \frac{T_L}{T_H}
 \end{aligned}$$

Here, in going to the second line we use the fact that the adiabatic paths BC and DA have no entropy change, and in the third line we are able to bring the temperatures out of the integral as they are constant over those paths. Also, note the critical negative sign: we've defined  $Q_L$  as the magnitude of heat exhausted, so the *heat absorbed* by the gas over CD is  $-Q_L$ . Plugging this ratio into Equation 1 give the desired result:

$$e = 1 - \frac{T_L}{T_H}$$

If you did not think to use this trick, it is still straightforward to compute  $Q_H$  and  $Q_L$  directly. By the First Law (Energy Conservation) the heat exchanged over the isothermal legs AB and CD is equal to the work done over those legs, and so we can compute the heat by integrating  $dW = P dV$ :

$$Q_H = W_H = \int_{V_A}^{V_B} P dV = NkT_H \int_{V_A}^{V_B} \frac{1}{V} dV = NkT_H \ln \left( \frac{V_B}{V_A} \right)$$

where  $N$  is the number of molecules in the gas,  $k$  is Boltzmann's constant, and we were able to bring  $T_H$  out of the integral since temperature is constant along AB. Similarly for  $Q_L$ :

$$\begin{aligned}
 -Q_L = W_L &= \int_{V_C}^{V_D} P dV = NkT_L \int_{V_C}^{V_D} \frac{1}{V} dV = NkT_L \ln \left( \frac{V_D}{V_C} \right) \\
 \Rightarrow Q_L &= NkT_L \ln \left( \frac{V_C}{V_D} \right) > 0
 \end{aligned}$$

Again we must be careful with signs: we've defined  $Q_L$  to be the magnitude of the output heat but the above integral computes the *heat absorbed*, so we must include an explicit minus sign on the left-hand side above. We now have, using Equation 1:

$$e = 1 - \frac{T_L \ln(V_C/V_D)}{T_H \ln(V_B/V_A)}$$

The various volumes  $V_A, V_B, V_C, V_D$  are not independent. We can choose the values of  $V_A < V_B < V_C$  as we wish, but once these are chosen, the value of  $V_D$  is uniquely determined as the point that connects both to  $V_A$  via an adiabat and to  $V_C$  via an isotherm. So we can simplify  $e$  above by writing  $V_D$  in terms of the

other three volumes. The volumes are related by the four paths:

$$\begin{aligned} P_A V_A &= P_B V_B & [A \rightarrow B \text{ isothermal}] \\ P_B V_B^\gamma &= P_C V_C^\gamma & [B \rightarrow C \text{ adiabatic}] \\ P_C V_C &= P_D V_D & [C \rightarrow D \text{ isothermal}] \\ P_D V_D^\gamma &= P_A V_A^\gamma & [D \rightarrow A \text{ adiabatic}] \end{aligned}$$

and we now solve these equations for  $V_D$  in terms of  $V_A, V_B, V_C$ . There are many paths through this arithmetic. A relatively clean one is to replace the pressures with temperatures using the ideal gas law, and then make use of the fact that  $T_A = T_B$  and  $T_C = T_D$  to simplify. But I will take the most direct route. From the bottom two equations eliminate  $P_D$ :

$$P_C V_C = V_D^{1-\gamma} P_A V_A^\gamma$$

Now use the second equation to eliminate  $P_C$  from this expression:

$$\begin{aligned} P_B V_B^\gamma V_C^{1-\gamma} &= V_D^{1-\gamma} P_A V_A^\gamma \\ \Rightarrow V_D^{1-\gamma} &= \frac{P_B V_B^\gamma V_C^{1-\gamma}}{P_A V_A^\gamma} \end{aligned}$$

And now we can eliminate the ratio of pressures in this expression using the first equation:

$$\begin{aligned} V_D^{1-\gamma} &= \frac{V_A^{1-\gamma} V_C^{1-\gamma}}{V_B^{1-\gamma}} \\ \Rightarrow \frac{V_C}{V_D} &= \frac{V_B}{V_A} \end{aligned}$$

We've found that in order for the Carnot cycle to form a closed loop, the volume compression ratio over each isothermal path must be equal. And so the ratio of logs appearing in our expression for the efficiency is just 1:

$$\frac{\ln(V_C/V_D)}{\ln(V_B/V_A)} = 1$$

and we find the efficiency of this engine to be:

$$e = 1 - \frac{T_L}{T_H}$$

## Rubric

### 3a [5 pts]

These criteria were judged independently and their points summed (a perfect answer satisfies all three).

2 pts: Correct ordering and relative locations of ABCD

2 pts: Correct shapes of paths connecting ABCD

1 pts: Drawing a closed, four-vertex figure in PV plane

### 3b [15 pts]

Since there are many possible routes to the correct efficiency (and many more possible missteps), this was judged on the soundness of your approach rather than specific features. Each answer was placed into one of the following categories with the corresponding score:

15 pts: Fully valid derivation.

12 pts: Valid derivation, but left unsimplified and does show explicit dependence of efficiency on only the temperature ratio.

8 pts: Fundamental approach was sound, but derivation invalid due to logical gaps or significant conceptual errors in calculations.

4 pts: An attempt to derive efficiency from first principles was made, but the approach was fundamentally unsound or too vague to be understood.

0 pts: No attempt at a derivation (including simply quoting the correct efficiency from memory).

Additionally, an extra  $-2$  points was assessed whenever one of the first three categories would have been met except for arithmetic errors.



## 4.

### (a)

After a molecule collides with the wall, it must travel the length of the cylinder and then travel back before again colliding with the same wall, for a total distance of  $2L$ . The molecule's speed (magnitude of velocity) is  $v_x$ , and time to travel a distance equals distance divided by speed.

$$\Delta t = \frac{\Delta x}{v} = \frac{2L}{v_x}$$

### (b)

During each collision, the change in the molecule's momentum is twice the initial momentum itself – because the collision not only stops the molecule's forward progress but also sends it traveling back with the same speed. Since  $F = \frac{dp}{dt}$ , the average force the wall must exert equals the momentum transfer in each collision per time interval between collisions. Finally, the total force from  $N$  molecules equals  $N$  times the average force of one molecule.

$$\begin{aligned}\Delta p &= mv_x - (-mv_x) = 2mv_x \\ F_{\text{average; one molecule}} &= \frac{\Delta p}{\Delta t} = \frac{(2mv_x)}{\left(\frac{2L}{v_x}\right)} = \frac{mv_x^2}{L} \\ F &= \frac{Nmv_x^2}{L}\end{aligned}$$

### (c)

We can use the given equation to express our force formula in terms of temperature.

$$\begin{aligned}T &= \frac{mv^2}{k_B} \implies k_B T = mv_x^2 \\ F &= \frac{N(k_B T)}{L}\end{aligned}$$

$$\boxed{FL = Nk_B T}$$

(The result resembles the usual three-dimensional ideal gas law, but with force and length serving as the one-dimensional analogues of pressure and volume, respectively.)

## Scoring

6 points for part (a)

7 points for part (b)

7 points for part (c)

# Birgenau Midterm 1 2016 Problem 5 Solution

James Reed Watson

September 29, 2016

## 1 Solution

- (a)  $Q = mc\Delta T$ . Because the solid (a) line rises in temperature faster as a function of heat, then the dotted line (b) has a higher specific heat.
- (b) The latent heat across a phase change is given by  $mL = Q$ , where  $m$  is the mass of the object and  $Q$  is the heat applied. Our mass is 1 kg, so we just subtract the starting and ending energies on the graph.

$$L_f = 165 \frac{J}{Kg} - 60 \frac{J}{Kg} = 105 \frac{J}{Kg} \quad (1)$$

$$L_v = 620 \frac{J}{Kg} - 270 \frac{J}{Kg} = 350 \frac{J}{Kg} \quad (2)$$

- (c)  $A$  and  $C$  are in mixed phases.  $A$  is the solid-liquid region.  $C$  is the liquid gas region.  $A$  could take a less time than  $C$  because  $L_f < L_v$  for all substances.
- (d) This is a calorimetry problem. The system is thermally isolated, so the heat lost by the water and container is gained by the mercury. The mercury gains heat while it melts, and later as it reaches the equilibrium

temperature of  $30^\circ\text{C}$ . The equation to use is

$$m_m c_m (T_f - T_0) + m_m L_f + m_{al} c_{al} (T_f - T_w) + c_w m_w (T_f - T_w) \quad (3)$$

$$L_f = -c_m (T_f - T_0) - \frac{m_{al}}{m_m} c_{al} (T_f - T_w) - \frac{m_w}{m_m} c_w (T_f - T_w) \quad (4)$$

$$= -1400 \frac{\text{J}}{\text{kgK}} (70\text{K}) - 2(900 \frac{\text{J}}{\text{kg}} \text{K})(-20\text{K}) - 1400 \frac{\text{J}}{\text{kg}} \text{K}(-20\text{K}) \quad (5)$$

$$= -98000 \frac{\text{J}}{\text{kg}} + 84000 \frac{\text{J}}{\text{kg}} + 36000 \frac{\text{J}}{\text{kg}} \quad (6)$$

$$= 22,000 \frac{\text{J}}{\text{kg}} \quad (7)$$

## 2 Rubric

My grading is negative, and if their answer is correct I don't plan on deducting points.

1. a) Answer incorrect (5.0)
2. b)  $L_f$  correct (2.0)
3. b)  $L_v$  correct (2.0)
4. b) Units correct (1.0)
5. c) Identified regions  $A$  and  $C$ . (1.0)
6. c) Identified  $A$  as solid liquid and  $C$  as liquid gas. (2.0)
7. c) Identified  $C$  as taking longer (1.0)
8. c) Identified the correct reason for  $C$  taking longer.(1.0)
9. d) Correct answer (2.5)
10. d) Wrote down  $\Delta Q_{total} = 0$ . (1.0)
11. d) Wrote down expression for  $L$ . (1.0)
12. d) Units correct (.5)

Total: 20 points, 5 points per problem.