Physics 7B Midterm 1 Problem 1 Solution

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(a) Suppose some amount of heat, Q, is added to the gas, while the volume is held constant. This is equivalent to adding some amount of heat to each of the two types of gas. Suppose the amount of heat added to the monatomic gas is Q_1, and the amount of heat added to the diatomic gas is Q_2. Since the total heat added is Q, we have that:

$$Q = Q_1 + Q_2. \tag{1}$$

Since the two types of gas are at thermal equilibrium before and after the addition of heat, the change in temperature of the two gases must be equal. This change in temperature is then also equal to the change in temperature of the mixture. Let this change in temperature be ΔT. Thus, since volume is held constant,

$$Q_1 = n_1 C_{V1} \Delta T, \tag{2}$$

and

$$Q_2 = n_2 C_{V2} \Delta T. \tag{3}$$

The specific heat at constant volume of the mixture would then be defined as:

$$\begin{aligned} C_V &= \frac{Q}{(n_1 + n_2)\Delta T} \\ &= \frac{Q_1 + Q_2}{(n_1 + n_2)\Delta T}, \end{aligned} \tag{4}$$

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where the second line follows from (1). Plugging equations (2) and (3) into this expression then gives the solution:

$$
\begin{aligned}
C_V &= \frac{n_1 C_{V1} \Delta T + n_2 C_{V2} \Delta T}{(n_1 + n_2)\Delta T} \\
&= \frac{n_1 C_{V1} + n_2 C_{V2}}{(n_1 + n_2)}.
\end{aligned}
\tag{5}
$$

(b) For a reversible adiabatic expansion of an ideal gas, we have that:

$$
TV^{\gamma - 1} = \text{constant},
\tag{6}
$$

where $\gamma = \frac{C_P}{C_V}$. For an ideal gas, the specific heat at constant pressure, C_P, is given by $C_P = C_V + R$. Thus, $\gamma = 1 + \frac{R}{C_V}$. From this expression for γ, it follows that:

$$
TV^{\frac{R}{C_V}} = \text{constant}.
\tag{7}
$$

Therefore, if the gas expands from initial temperature and volume T_1 and V_1, to final temperature and volume T_f and V_f, we have that:

$$
T_f V_f^{\frac{R}{C_V}} = T_1 V_1^{\frac{R}{C_V}}.
\tag{8}
$$

Solving for T_f then gives:

$$
T_f = T_1 \left(\frac{V_1}{V_f} \right)^{\frac{R}{C_V}}.
\tag{9}
$$

We are given that $V_f = 2V_1$. Thus,

$$
T_f = T_1 2^{-\frac{R}{C_V}}.
\tag{10}
$$

Therefore, using our expression from part (a),
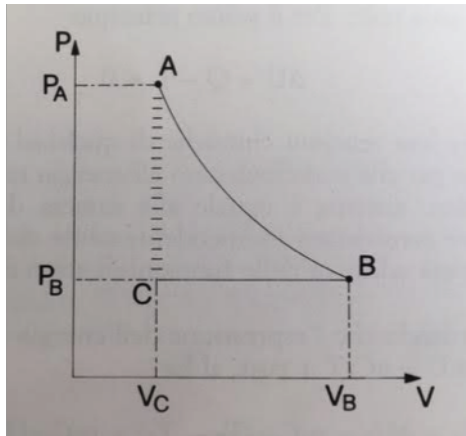
$$
T_f = T_1 2^{-\frac{(n_1 + n_2)R}{n_1 C_{V1} + n_2 C_{V2}}}.
\tag{11}
$$

Lanzara Midterm 1, Question 2 Rubric

a.

Solution:

Isotherm from $V_a = V_c$ to $V_b = 2V_c$; Isobarametric from V_b to $V_c = V_a$. Technically, V_c to V_a cannot be drawn on a PV diagram because it is irreversible, so this process should be illustrated as some sort of dashed/ incomplete line. See graph below:

Point Allocation:

Isotherm: 2pts
Isobarametric: 2pts (must lead back to correct volume)
Isovolumetric: 1pt (represented by some sort of dashed line)

b.

Solution:

This problem can be divided up into several steps. The first is the isotherm:

$$W_{ab} = \int_{V_a}^{V_b} pdV = nRT_a \int_{V_a}^{V_b} \frac{dV}{V} = nRT_a ln\frac{V_b}{V_a} \tag{1}$$

Then we have the isobaric:

$$W_{bc} = \int_{V_b}^{V_c} p_b dV = p_b \int_{V_b}^{V_c} dV = p_b(V_c - V_b) \tag{2}$$

We then realize that, as there is no change in volume, no work is done during the isovolumetric process. We can now compute the total work done as the sum of the work done during each path:

$$W_{total} = W_{ab} + W_{bc} = nRT_a ln\frac{V_b}{V_a} + p_b(V_c - V_b) \tag{3}$$

This can be simplified using the relations given and the ideal gas law:

$$W_{total} = p_b V_b ln2 + \frac{1}{2} p_b V_b - p_b V_b = p_b V_b ln2 - \frac{1}{2} p_b V_b \tag{4}$$

There are multiple acceptable forms to express the work, but in this form, knowing that $ln2$ is greater than $1/2$, it is obvious that the work done during the cycle is positive. It is also acceptable to use a pV diagram with knowledge of the direction of the cycle to claim that the work done by the gas is positive.

Point Allocation:
Isothermal work: 3 pts
Isobarametric work: 2 pts
No isovolumetric work: 1 pt
Work total as the sum: 1pt
Work done is positive with some reasoning: 1 pt

Solution:
This problem can again be divided into three segments. First is the isotherm, where $\Delta E = 0 = Q - W$. Thus:

$$Q = W = nRT_a ln2 = RT_a ln2 \tag{5}$$

We can then focus on the isobarametric process:

$$Q = nC_p \Delta T = nC_p(T_c - T_a) = \frac{C_p p_b}{R}(V_c - V_b) = -\frac{C_p p_b}{R} V_c = -\frac{C_p p_b}{R} \tag{6}$$

Where the last few steps were simplified using the fact that $V = 1m^3$. Finally, we have the isvolumetric process:

$$Q = nC_v \Delta T = nC_v(T_a - T_c) = nC_v(T_a - \frac{p_b V_c}{nR}) = nC_v(T_a - \frac{p_b}{nR}) \tag{7}$$

It's not entirely clear what variables could be considered known, and what were not. For that reason, I will allow any variable except T_c. Volumes should be evaluated using $V_c = 1$

Point Allocation:
Isothermal heat: 2 pt
Isobarametric heat: 2 pt
Isovolumetric heat: 2 pt
One of the following: 1 pt
-A simplified solution not in terms T_c or a numeric volume
-Some discussion of $C_v = \frac{3}{2}R$ and $C_p = \frac{5}{2}R$

3.

1)   Because the total system of water+ice is thermally isolated, the change in internal energy is zero.

   5 pts for recognizing that system is thermally isolated so energy is zero (all or nothing)

2)   Net heat exchanged must be zero because system is thermally isolated. So:

$$Q = Q_{ice} + Q_W = 0$$

$$Q_{ice} = ML + MC_W (T_{final} - T_{ice})$$

$$Q_W = MC_W (T_{final} - T_W)$$

Since $Q_{ice} = -Q_W$ and T_{ice} = 0 C:

$$T_{final} = \frac{1}{2C_W} (C_W T_W - L)$$

   5 pts for setting up problem correctly (i.e. net Q = 0 = sum of heat terms). Partial allotment of points if they got some terms correct but missed others

   5 pts for working through and getting correct answer. **2/5 pts** if they set it up incorrectly but otherwise made no errors. **3/5 pts** if they set it up correctly but made an algebraic mistake.

Fall 2016 - Physics 7B Sections 2 & 3 Midterm 1 - Q4 Solution

Question:

Two identical containers contain two different diatomic gases. While the total mass of the gas in each container is the same, the total number of molecules in A is N_A at pressure P_A, and the number of molecules in B is N_B at pressure P_B. The two gases are at the same temperature T.

(a) (5pts) What is the ratio of RMS velocities, of the gas in A and B?

(b) (10pts) If we want $v_{RMS}^A = v_{RMS}^B$, by what fraction $(T_A - T)/T$ should the new temperature T_A, in container A be changed from T?

Solution:

(a) (5pts)

Definition of v_{RMS} **(1pt)**:

$$v_{RMS} = \sqrt{\frac{3k_B T}{m}}$$

With T, the temperature of the gas and m, the mass per gas molecule.

Let the total mass of gas in each container be M. **(1pt)** Define the molecular mass of $m_A = M/N_A$ and $m_B = M/N_B$.

The ratio of the two gases is:

$$\frac{v_{RMS}^A}{v_{RMS}^B} = \sqrt{\frac{\frac{3k_B T}{m_A}}{\frac{3k_B T}{m_B}}} = \sqrt{\frac{m_B}{m_A}}$$

$$= \sqrt{\frac{M/N_B}{M/N_A}}$$

$$= \sqrt{\frac{N_A}{N_B}}$$

(2pts) - set up and propagate ratio

(1pt) - final answer: $\sqrt{\frac{N_A}{N_B}}$ or equivalently $\sqrt{\frac{P_A}{P_B}}$

(b) (10pts)

Temperature in B stays at T, while temperature in A changes to T_A

Condition on $v_{RMS}^A = v_{RMS}^B$ means **(2pts)**:

$$\sqrt{\frac{3k_B T_A}{m_A}} = \sqrt{\frac{3k_B T}{m_B}}$$

Solve for T_A **(3pts)**:

$$\frac{3k_B T_A}{m_A} = \frac{3k_B T}{m_B} \implies T_A = \frac{m_A}{m_B} T$$

Write in terms of variables given in the question:

$$T_A = \frac{N_B}{N_A} T$$

Fractional change algebra **(3pts)** and final answer **(2pts)** :

$$\frac{T_A - T}{T} = \frac{\frac{N_B}{N_A} T - T}{T}$$

$$= \frac{N_B}{N_A} - 1$$

1

Rubric

(a) 5pts

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    (1pt) - Definition of v_{RMS} with correct dependence on T and m or derivation from the kinetic theory definition of temperature.

    (1pt) - Realisation that the molecular masses of the two gases are different and given by $m_A = M/N_A$, $m_B = M/N_B$

    (2pts) - Writing down and attempting to simplify expression for the ratio of v_{RMS}^A/v_{RMS}^B

    (1pt) - Correct final answer: $\sqrt{\frac{N_A}{N_B}}$ or equivalently $\sqrt{\frac{P_A}{P_B}}$

(b) 10pts

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    2pts - Correct equality of v_{RMS} with new temperature T_A in v_{RMS}^A.

    3pts - Solve for an expression for T_A in terms of T.

    3pts - Writing down an expression for the fractional change and cancelling out T.

    2pts - Correct final answer $\frac{N_B}{N_A} - 1$

5.

Part I

In a free expansion Q=0, W=0 and therefore the internal energy change is equal to zero. The latter implies $\Delta T = 0$.

It is wrong to state, from $\Delta S = \int dQ/T$ that the change of entropy is zero, because this expression is valid only for a reversible process.
A free expansion is not a reversible process.

We need to find a path that connects the initial and final state. The simplest path is the isotherm.

$$\Delta S = \int \frac{dQ}{T} = \frac{Q}{T}$$

For an isotherm $Q = W = nrT \ln \frac{V_{final}}{V_{initial}} = nRln2 > 0$

3 pts for recognizing that in a free expansion, Q = W = 0, and temperature is constant.

3 pts for recognizing that process is irreversible and so we can't use the entropy equation. **2/3 pts** if they don't and use the equation to calculate a wrong answer (e.g. entropy is constant)

4 pts for substituting an appropriate reversible reaction (e.g. isotherm) to calculate the sign of entropy change. **2/4 pts** if they get the right answer but don't show their work. **1/4 pts** if they do the right method but get a wrong answer due to a mistake.

Problem 5

Part 2

All approaches to this can be traced to the second law in various forms. An example of this is that any engine can have at best the Carnot efficiency and that to have a perfect efficiency with a Carnot engine requires exhausting to zero temperature which is an unobtainable setting. Another possible approach is stating that entropy must be non decreasing but the change in entropy of this system and its surroundings would be less than zero, argued as follows; $\Delta S = \Delta S_{gas} + \Delta S_{reservoir} = 0 - \frac{Q}{T_H} < 0$.

For this problem the answers were highly variable, 5 points were awarded for a correct statement of the second law and the remaining 5 points were awarded for a correct argument following from the first statement that showed why this wasn't possible.

Part 3

The efficiency of our cycle is $\frac{W}{Q_H}$ which by conservation of energy is $\frac{Q_H - Q_L}{Q_H}$ so to calculate the efficiency we can either calculate the total work or the heat input/exhaust. No heat exchange occurs during an adiabatic process. During the isothermal process, by the first law $\Delta E = 0 = Q - W$. So the heat is equal to the work which is positive (this is our input). Conservation of energy then dictates that the heat in the isobaric process must be negative so this is Q_L.

Putting this together $Q_H = nRT_B ln(\frac{V_C}{V_B})$ and $Q_L = nC_P(T_C - T_A)$. Now all that is left to do is to write these expressions in terms of P_0 and T_0.

Because A and B are related by adiabatic compression $P_0 V_A^\gamma = (2P_0)V_B^\gamma$, so $V_B = 2^{-\frac{1}{\gamma}}V_A$. Replacing V using the ideal gas law on both sides this tells us $T_B = 2^{1-\frac{1}{\gamma}}T_0$ (used $T_A = T_0$ and $P_B = 2P_0$). Additionally we know that $\frac{V_C}{V_B} = \frac{\frac{nRT_C}{P_C}}{\frac{nRT_B}{P_B}}$. B and C are connected by an isotherm so $T_C = T_B$. $P_B = 2P_0$ and $P_C = P_0$ because the isobaric process closes the cycle to A. Together this means $\frac{V_C}{V_B} = 2$. $\gamma = \frac{5/2R}{3/2R} = \frac{5}{3}$ for a monatomic ideal gas.

Substituting the results of the last paragraph: $e = \frac{nR2^{2/5}T_0 ln(2) - nR\frac{5}{2}T_0(2^{2/5}-1)}{nR2^{2/5}T_0 ln(2)} = \frac{2^{2/5}ln(2) - \frac{5}{2}(2^{2/5}-1)}{2^{2/5}ln(2)}$

3 points were awarded to each of the following components; the correct drawing of the cycle in P-V space, the calculation of W, the calculation of Q_H and one point was given to the calculation of e (not just writing down the definition). If W and Q_H were not completed correctly partial credit was assigned. Writing down the statements $Q_H = nRT_B ln(\frac{V_C}{V_B})$ and $Q_L = nC_P(T_C - T_A)$ were worth two points each. Showing $\frac{V_C}{V_B} = 2$ and $T_B = 2^{2/5}T_0$ were worth one point each.