Spring 2016 Physics 7A Lec 001 (Yildiz) Midterm II

1. (15 points) An object is released from rest at an altitude *h* above the surface of the Earth. *h* is comparable to the radius of Earth (R_E), so gravitational acceleration (g) is not constant.

a) What is the velocity of the object when it hits the surface of Earth?

b) What is the gravitational acceleration of the object at distance *r* from the Earth's center, where $R_E < r < R_E + h$?

c) What is the rate of change of the gravitational acceleration $g(r)$ as a function of the distance *r* from the Earth's center, where $R_E < r < R_E + h$?

Solution:

a) $-\Delta U = \Delta K$

$$U_G(r) = -\frac{GMm}{r}$$

$$\frac{1}{2}mv^2 = -GMm\left(\frac{1}{R_E + h} - \frac{1}{R_E}\right)$$

$$v = 2GM\left(\frac{1}{R_E} - \frac{1}{R_E+h}\right)$$

b) $F_G(r) = -\frac{GMm}{r^2} = m.g(r)$

$$g(r) = -\frac{GM}{r^2}$$

c) $\frac{dg}{dr} = \frac{2GM}{r^3}$

2. (20 points) Assume a cyclist of weight *mg* can exert a force on the pedals equal to 0.80 *mg* on the average. The pedals rotate in a circle of radius 18 cm, the wheels have a radius of 34 cm, and the front and back sprockets on which the chain runs have 42 and 19 teeth respectively. The mass of the bike is 14 kg and that of the rider is 63 kg. Assume there is no slipping between the ground and the wheel. Use g = 10 m/s^2 for calculations.

a) How is the angular velocity of the rear wheel of a bicycle related to the angular velocity of the front sprocket and pedals? The teeth are spaced the same on both sprockets and the rear sprocket is firmly attached to the rear wheel.

b) Determine the maximum steepness of hill the cyclist can climb at constant speed. Assume the cyclist's average force is always *tangential* to pedal motion.

c) Determine the maximum steepness of hill the cyclist can climb at constant speed. Assume the cyclist's average force is always *downward*.

If the rider is riding at a constant speed, then the positive work input by the rider to the (bicycle + rider) combination must be equal to the negative work done by gravity as he moves up the incline. The net work must be 0 if there is no change in kinetic energy.

(*a*) If the rider's force is directed downwards, then the rider will do an amount of work equal to the force times the distance parallel to the force. The distance parallel to the downward force would be the diameter of the circle in which the pedals move. Then consider that by using 2 feet, the rider does twice that amount of work when the pedals make one complete revolution. So in one revolution of the pedals, the rider does the work calculated below.

$$W_{\text{rider}} = 2\left(0.90 m_{\text{rider}} g\right) d_{\substack{\text{pedal}\\\text{motion}}}$$

In one revolution of the front sprocket, the rear sprocket will make $42/19$ revolutions, and so the back wheel (and the entire bicycle and rider as well) will move a distance of $\left(42/19\right)\left(2\pi r_{\text{wheel}}\right)$. That is a distance along the plane, and so the height that the bicycle and rider will move is $h = \left(42/19\right)\left(2\pi r_{\text{wheel}}\right)\sin\theta$. Finally, the work done by gravity in moving that height is calculated.

$$W_{\text{G}} = \left(m_{\text{rider}} + m_{\text{bike}}\right) gh\cos 180^\circ = -\left(m_{\text{rider}} + m_{\text{bike}}\right) gh = -\left(m_{\text{rider}} + m_{\text{bike}}\right) g\left(42/19\right)\left(2\pi r_{\text{wheel}}\right)\sin\theta$$

Set the total work equal to 0, and solve for the angle of the incline.

$$W_{\text{rider}} + W_{\text{G}} = 0 \;\rightarrow\; 2\left[0.90 m_{\text{rider}} g\right] d_{\substack{\text{pedal}\\\text{motion}}} - \left(m_{\text{rider}} + m_{\text{bike}}\right) g\left(42/19\right)\left(2\pi r_{\text{wheel}}\right)\sin\theta = 0 \;\rightarrow$$

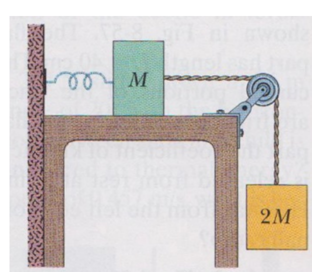
$$\theta = \sin^{-1}\frac{\left(0.90 m_{\text{rider}}\right) d_{\substack{\text{pedal}\\\text{motion}}}}{\left(m_{\text{rider}} + m_{\text{bike}}\right)\left(42/19\right)\left(\pi r_{\text{wheel}}\right)} = \sin^{-1}\frac{0.90\left(65\,\text{kg}\right)\left(0.36\,\text{m}\right)}{\left(77\,\text{kg}\right)\left(42/19\right)\pi\left(0.34\,\text{m}\right)} = \boxed{6.7^\circ}$$

(*b*) If the force is tangential to the pedal motion, then the distance that one foot moves while exerting a force is now half of the circumference of the circle in which the pedals move. The rest of the analysis is the same.

$$W_{\text{rider}} = 2\left(0.90 m_{\text{rider}} g\right)\left(\pi r_{\substack{\text{pedal}\\\text{motion}}}\right) \;;\; W_{\text{rider}} + W_{\text{G}} = 0 \;\rightarrow$$

$$\theta = \sin^{-1}\frac{\left(0.90 m_{\text{rider}}\right)\pi r_{\substack{\text{pedal}\\\text{motion}}}}{\left(m_{\text{rider}} + m_{\text{bike}}\right)\left(42/19\right)\left(\pi r_{\text{wheel}}\right)} = \sin^{-1}\frac{0.90\left(65\,\text{kg}\right)\left(0.18\,\text{m}\right)}{\left(77\,\text{kg}\right)\left(42/19\right)\left(0.34\,\text{m}\right)} = 10.5^\circ \approx \boxed{10^\circ}$$

3. (20 points) Two block of masses *M* and *2M* are connected to a spring of spring constant *k* that has one end fixed, as shown. The horizontal surface



and the pulley are frictionless and the pulley has negligible mass. The blocks are released from rest with the spring relaxed.

a) What is the velocity of the blocks when the hanging block has fallen a distance *h*?

b) What maximum distance h_{max} does the hanging block fall before momentarily stopping?

c) In the absence of friction, the system is expected to oscillate back and forth infinitely. However, if there is kinetic friction between block and the horizontal surface (coefficient of kinetic friction is μ_k), the system will eventually come to a complete stop. What is the equilibrium position of the spring's extension when the system comes to a complete stop? What is the total distance traveled by the hanging block before the system comes to a complete stop? (Assume that the frictional force on the mass *M* on the horizontal surface is negligible when the system comes to a complete stop.)

Solution

a) $U_i + K_i = U_f + K_f$

$$0 = \left(\frac{1}{2}kh^2 - 2Mgh\right) + \left(\frac{1}{2}Mv^2 + \frac{1}{2}2Mv^2\right)$$

$$v = \sqrt{\frac{4Mgh - kh^2}{3M}}$$

b) $U_i + K_i = U_f + K_f$ and $v_f = 0 \ when\ h = h_{max}$

$$0 = \left(\frac{1}{2}kh_{max}{}^2 - 2Mgh_{max}\right) + 0$$

$$h_{max} = \frac{4Mg}{k}$$

c) In the presence of a non-conservative force, $U_i + K_i + W_{NC} = U_f + K_f$

$$W_{NC} = -F_{fr}d_{tot}$$

$$F_{fr} = \mu_k . N = \mu_k mg$$

$$K_f = 0\ (system\ is\ stationary), to\ find\ U_f, we\ need\ to\ find\ the\ equilibrium\ position.$$

$$At\ equilibrium, \sum F_x = 0, \qquad therefore\ \ T - kx_{eq} = 0 \quad (T\ is\ the\ tension\ on\ the\ rope)$$

$$Because\ the\ hanging\ mass\ is\ also\ stationary, \sum F_y = 0, \qquad therefore\ \ T - 2Mg = 0$$

$$2Mg = kx_{eq}$$

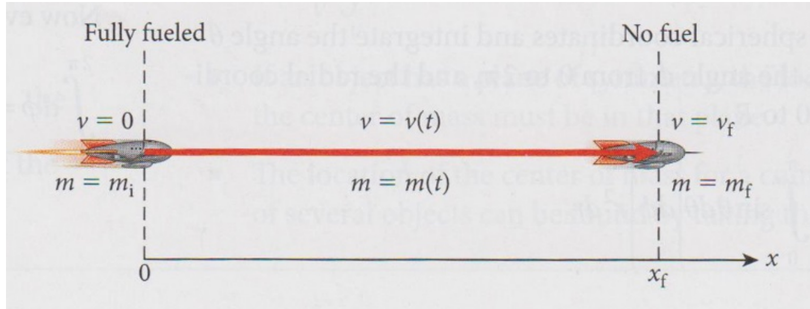
$$U_i + K_i + W_{NC} = U_f + K_f$$

$$0 - \mu_k mgd_{tot} = \left(\frac{1}{2}kx_{eq}{}^2 - 2Mgx_{eq}\right) + 0$$

$$-\mu_k mg d_{tot} = \frac{1}{2}k\left(\frac{2Mg}{k}\right)^2 - 2Mg\left(\frac{2Mg}{k}\right)$$

$$d_{tot} = \frac{2mg}{k\mu_k}$$

4. (25 points) Suppose a spacecraft with initial mass of m_i. Without its propellant, the spacecraft has a mass of $m_f = m_i/3$. The rocket that powers the spacecraft is designed to eject the propellant with a speed of u relative to the rocket at a constant rate of R. The spacecraft is initially at rest in space and travels in a straight line.



a) How long would it take the rocket to release all of its propellant?

b) What is $m(t)$, the mass of the rocket as a function of time?

c) What is $v(t)$, the speed of the rocket as a function of time?

d) How far will the spacecraft travel before its rocket uses all the propellant and shuts down?

Hint: $\int \ln(1 - ax)\, dx = \frac{ax-1}{a}\ln(1 - ax) - x$

Solution:

a) $t_f = \frac{m_i - m_f}{R} = \frac{m_i - \frac{m_i}{3}}{R} = \frac{2}{3}\frac{m_i}{R}$

b) $m(t) = m_i - Rt$

c) Because $\sum \boldsymbol{F}_{ext} = 0 \qquad m\frac{dv}{dt} = v_{rel}\frac{dm}{dt}$

$$\frac{dm}{dt} = -R \qquad v_{rel} = -u\ (regardless\ of\ the\ speed\ of\ the\ rocket)$$

$$(m_i - Rt)\frac{dv}{dt} = uR$$

$$v = \int_0^t \frac{uR}{(m_i - Rt)}\, dt$$

$$v = u ln\left(\frac{m_i}{m_i - Rt}\right)$$

d) $x_f = \int_o^{t_f} v dt = -u\int_o^{t_f} ln\left(1 - \frac{Rt}{m_i}\right) dt$

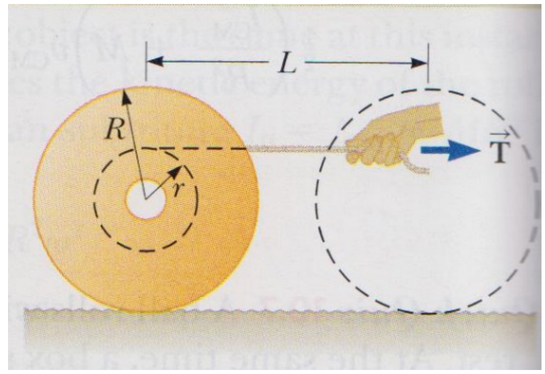
Using the Hint given in the question, $x_f = -u\left[\left(t - \frac{m_i}{R}\right) ln\left(1 - \frac{Rt}{m_i}\right) - t\right]_0^{t_f}$ and $t_f = \frac{2}{3}\frac{m_i}{R}$

$$x_f = -u\left[\left[\left(\frac{2}{3}\frac{m_i}{R} - \frac{m_i}{R}\right) ln\left(1 - \frac{R}{m_i}\frac{2}{3}\frac{m_i}{R}\right) - \frac{2}{3}\frac{m_i}{R}\right] - \left[-\frac{m_i}{R} ln\left(1 - \frac{R.0}{m_i}\right) - 0\right]\right]$$

$$x_f = -u\left[\left(\frac{1}{3}\frac{m_i}{R}\right)ln\left(1-\frac{2}{3}\right)-\frac{2}{3}\frac{m_i}{R}\right]$$

$$x_f = \frac{um_i}{R}\left[\frac{2}{3}-\frac{1}{3}ln3\right]$$

5. (20 points) A cylindrically symmetric spool of mass m and radius R sits at rest on a horizontal table with friction. With your hand on a massless string wrapped around the axle of radius r, you pull on the spool with a constant horizontal force of magnitude T to the right. As a result, the spool rolls without slipping a distance L along the table. Assume that the spool is a solid uniform cylinder.



a) What is the distance that your hand travels as the spool moves a distance L?

b) Find the final translational speed of the center of mass of the spool using the work-energy principle?

c) Find the value of the friction force f and acceleration of the spool using the equations for dynamics of the motion (e.g. force and torque)?

a) We first find the length of string that has unwound off the spool. If the spool rolls through a distance L, the total angle through which it rotates is $\theta = L/R$. The axle also rotates through this angle.

$$\ell = r\theta = \frac{r}{R}L$$

This result also gives the length of string pulled off the axle. Your hand will move through this distance *plus* the distance L through which the spool moves. Therefore, the magnitude of the displacement of the point of application of the force applied by your hand is $\ell + L = L(1 + r/R)$.

b)

$$(1)\quad W = \Delta K = \Delta K_{\text{trans}} + \Delta K_{\text{rot}}$$

Evaluate the work done by your hand on the string:

$$(2)\quad W = TL\left(1 + \frac{r}{R}\right)$$

Substitute Equation (2) into Equation (1):

$$TL\left(1 + \frac{r}{R}\right) = \tfrac{1}{2}mv_{\text{CM}}^{2} + \tfrac{1}{2}I\omega^2$$

where I is the moment of inertia of the spool about its center of mass and v_{CM} and ω are the final values after the wheel rolls through the distance L.

Apply the nonslip rolling condition $\omega = v_{\text{CM}}/R$:

$$TL\left(1 + \frac{r}{R}\right) = \tfrac{1}{2}mv_{\text{CM}}^{2} + \tfrac{1}{2}I\frac{v_{\text{CM}}^{2}}{R^2}$$

Solve for v_{CM}:

$$(3)\quad v_{\text{CM}} = \sqrt{\frac{2TL(1 + r/R)}{m(1 + I/mR^2)}}$$

c. In lectures, we used F = ma and $\tau = I\alpha$ to find frictional force and acceleration. Here is an alternative solution using Impulse-momentum theorem.

Categorize Because the friction force does no work, we cannot evaluate it from an energy approach. We model the spool as a nonisolated system, but this time in terms of momentum. The string applies a force across the boundary of the system, resulting in an impulse on the system. Because the forces on the spool are constant, we can model the spool's center of mass as a particle under constant acceleration.

Analyze Write the impulse–momentum theorem for the spool:

$$(4)\quad (T - f)\Delta t = m(v_{CM} - 0) = mv_{CM}$$

For a particle under constant acceleration starting from rest, the average velocity of the center of mass is half the final velocity.

find the time interval for the center of mass of the spool to move a distance L from rest to a final speed v_{CM}:

$$(5)\quad \Delta t = \frac{L}{v_{CM,avg}} = \frac{2L}{v_{CM}}$$

Substitute Equation (5) into Equation (4):

$$(T - f)\frac{2L}{v_{CM}} = mv_{CM}$$

Solve for the friction force f:

$$f = T - \frac{mv_{CM}^2}{2L}$$

Substitute v_{CM} from Equation (3):

$$f = T - \frac{m}{2L}\left[\frac{2TL(1 + r/R)}{m(1 + I/mR^2)}\right]$$

$$= T - T\frac{(1 + r/R)}{(1 + I/mR^2)} = T\left[\frac{I - mrR}{I + mR^2}\right]$$

Finalize Notice that we could use the impulse–momentum theorem for the translational motion of the spool while ignoring that the spool is rotating! This fact demonstrates the power of our growing list of approaches to solving problems.