MATH 185-4 MIDTERM 1 SOLUTION

1. (5 points) Determine whether the following statements are true of false, no justification is required.

(1) (1 point) The principal branch of logarithm function $f(z) = \text{Log}z$ is continuous on $\mathbb{C} \setminus \{0\}$.

False. Recall that $\text{Arg}(z) \in (-\pi, \pi]$. So $\text{Log}(-1) = \pi i$, and for $z_n = e^{i(\frac{1}{n}-1)\pi}$, we have $\lim z_n = e^{-i\pi} = -1$ and $\lim f(z_n) = \lim i(\frac{1}{n} - 1)\pi = -\pi i \neq f(-1)$.

(2) (1 point) Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers in a bounded set $U \subset \mathbb{C}$, then $\{z_n\}_{n=1}^{\infty}$ always has a subsequence converging to a point $z_0 \in U$.

False. The theorem need the condition that U is closed. For example, for $U = \{z \in \mathbb{C} \mid |z| < 1\}$ and $z_n = 1 - \frac{1}{n} \in U$, $\lim z_n = 1 \notin U$.

(3) (1 point) If $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ are both analytic functions, then $f \circ g : \mathbb{C} \to \mathbb{C}$ is also analytic.

True. This is a theorem on the book, and the chain rule gives $(f \circ g)'(z) = f'(g(z)) \cdot g'(z)$.

(4) (1 point) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function. If $f'(z_0) \neq 0$, then there exists a small open disc B containing z_0 such that f is one-to-one on B.

True. This is part of the inverse function theorem.

(5) (1 point) Let D be an open set in $\mathbb{C}$, any harmonic function $u : D \to \mathbb{R}$ has a harmonic conjugation.

False. For $D = \mathbb{C} \setminus \{0\}$, and $u(z) = \log |z|$, the harmonic conjugation does not exist on D. It only exists, for example, on $\mathbb{C} \setminus (-\infty, 0]$, and a harmonic conjugation is $v(z) = \text{Arg}z$.

2. (6 points) Please give the definitions of the following concepts.

(1) (2 points) $f : \mathbb{C} \to \mathbb{C}$ is **complex differentiable** at $z_0 \in \mathbb{C}$.

The limit
$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$
exists.

(2) (2 points) $f : \mathbb{C} \to \mathbb{C}$ is an **analytic function**.

For any $z \in \mathbb{C}$, the complex derivative $f'(z)$ exists, and the function $f' : \mathbb{C} \to \mathbb{C}$ is continuous.

(3) (2 points) The **Cauchy–Riemann equations** for $f(x + iy) = u(x, y) + iv(x, y)$.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

3. (6 points) Please do the following computations.

(1) (2 points) Please compute

$$\frac{5+5i}{3+4i} = ?$$

$$\frac{5+5i}{3+4i} = \frac{(5+5i)(3-4i)}{(3+4i)(3-4i)} = \frac{15+20+15i-20i}{25} = \frac{7-i}{5}.$$

(2) (2 points) Please find all complex numbers $z \in \mathbb{C}$ such that

$$(z-1)^4 = 8\sqrt{2} + 8\sqrt{2}i.$$

Since $8\sqrt{2} + 8\sqrt{2}i = 16(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) = 2^4 e^{i\frac{\pi}{4}}$, all complex numbers w such that $w^4 = 8\sqrt{2} + 8\sqrt{2}i$ are $2e^{i\frac{1}{16}\pi}$, $2e^{i\frac{9}{16}\pi}$, $2e^{i\frac{17}{16}\pi}$ and $2e^{i\frac{25}{16}\pi}$.

So all the desired complex numbers z are $2e^{i\frac{1}{16}\pi} + 1$, $2e^{i\frac{9}{16}\pi} + 1$, $2e^{i\frac{17}{16}\pi} + 1$ and $2e^{i\frac{25}{16}\pi} + 1$.

(3) (2 points) Please find all complex numbers $z \in \mathbb{C}$ such that

$$e^{z+\frac{\pi}{6}i} = 1 + \sqrt{3}i.$$

Since $1 + \sqrt{3}i = 2(\frac{1}{2} + \frac{\sqrt{3}}{2}i) = 2e^{i\frac{\pi}{3}}$, $e^{z+\frac{\pi}{6}i} = 1 + \sqrt{3}i = 2e^{i\frac{\pi}{3}}$ if and only if $e^z = 2e^{i\frac{\pi}{6}}$.

So all the possible z are $\log 2 + \frac{\pi}{6}i + 2k\pi i$, with $k \in \mathbb{Z}$.

4. (8 points) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function, and suppose that $f(x+iy) = u(x,y) + iv(x,y)$. If all second-order partial derivatives of u and v are continuous, please show that the complex derivative $f' : \mathbb{C} \to \mathbb{C}$ is also analytic.

Since f is analytic, the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

hold on $\mathbb{C}$. Moreover, we also have

$$f'(x+iy) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y).$$

To check that f' is analytic, we need to check the expression of f' above satisfies the Cauchy-Riemann equations.

For the first equation, we check it by

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right).$$

Here the second equality uses the fact that all second-order partial derivatives of v are continuous.

Similarly, for the second equation, we also check it by

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right).$$

So the Cauchy-Riemann equations hold for f', this implies that $f''(z)$ exists for any $z \in \mathbb{C}$ and

$$f''(x+iy) = \frac{\partial^2 u}{\partial x^2}(x,y) + i\frac{\partial^2 v}{\partial x^2}(x,y).$$

Moreover, since all second-order partial derivatives of u and v are continuous, f'' is continuous. So we have showed that $f' : \mathbb{C} \to \mathbb{C}$ is analytic.

5. (8 points) Let $u : \mathbb{C} \to \mathbb{R}$ be a harmonic function, and let $v : \mathbb{C} \to \mathbb{R}$ be its harmonic conjugation.

(1) (3 points) Let $p : \mathbb{C} \to \mathbb{R}$ be a function defined by $p(x, y) = u(-y, x) + v(x, y)$, please show that p is a harmonic function.

Since u is harmonic, and v is the harmonic conjugation of u, both u and v are harmonic functions. So

$$u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$$

hold on $\mathbb{C}$.

Now we do computations for p.

$$p_x(x, y) = u_y(-y, x) + v_x(x, y), \text{ and } p_{xx}(x, y) = u_{yy}(-y, x) + v_{xx}(x, y).$$

Similarly,

$$p_y(x, y) = -u_x(-y, x) + v_y(x, y), \text{ and } p_{yy}(x, y) = u_{xx}(-y, x) + v_{yy}(x, y).$$

So

$$p_{xx}(x, y) + p_{yy}(x, y) = \big(u_{xx}(-y, x) + u_{yy}(-y, x)\big) + \big(v_{xx}(x, y) + u_{yy}(x, y)\big) = 0.$$

Moreover, since all first-order and second-order partial derivatives of p are linear combinations of first-order and second-order partial derivatives of u and v, they are all continuous. So p is a harmonic function on $\mathbb{C}$.

(2) (5 points) Please find a harmonic conjugation $q : \mathbb{C} \to \mathbb{R}$ of p.

Since v is the harmonic conjugation of u, $f(x + iy) = u(x, y) + iv(x, y)$ is an analytic function. We need to find a function q such that $p(x, y) + iq(x, y)$ is also analytic.

For $g : \mathbb{C} \to \mathbb{C}$ defined by $g(z) = f(iz) - if(z)$, g is apparently an analytic function. We also have

$$\begin{aligned} g(x + iy) &= f(-y + ix) - if(x + iy) = \big(u(-y, x) + iv(-y, x)\big) - i\big(u(x, y) + iv(x, y)\big) \\ &= \big(u(-y, x) + v(x, y)\big) + i\big(v(-y, x) - u(x, y)\big). \end{aligned}$$

Since g is analytic and p is the real part of g, the imaginary part $q(x, y) = v(-y, x) - u(x, y)$ is the harmonic conjugation of p.

6. (7 points)

(1) (4 points) Please find the fractional linear transformation that maps 0 to $2i$, 1 to 0, $\{z \in \mathbb{C} \mid |z| = 1\}$ to $\{x + iy \in \mathbb{C} \mid x = y\}$, and $\{x + iy \in \mathbb{C} \mid y = 0\}$ to $\{z \in \mathbb{C} \mid |z - (1 + i)| = \sqrt{2}\}$.

Let f be the desired fractional linear transformation.

The intersection between $\{z \in \mathbb{C} \mid |z| = 1\}$ and $\{x + iy \in \mathbb{C} \mid y = 0\}$ is $\{1, -1\}$, and the intersection between $\{x + iy \in \mathbb{C} \mid x = y\}$ and $\{z \in \mathbb{C} \mid |z - (1 + i)| = \sqrt{2}\}$ is $\{0, 2 + 2i\}$. So f maps $\{1, -1\}$ to $\{0, 2 + 2i\}$. Since we already know that f maps 1 to 0, it maps -1 to $2 + 2i$.

Since $f(1) = 0$, we have

$$f(z) = \frac{a(z-1)}{cz+d}.$$

Since $f(0) = 2i$, we further get

$$f(z) = \frac{2i(z-1)}{cz-1}.$$

Since we also have $f(-1) = 2+2i$, we have $\frac{-4i}{-c-1} = 2+2i$. So $c = \frac{4i}{2+2i} - 1 = (1 + i) - 1 = i$, and the desired fractional linear transformation is

$$f(z) = \frac{2i(z-1)}{iz-1} = \frac{2z-2}{z+i}.$$

(2) (3 points) Please also find the image of the line $\{x + iy \mid x = 0\}$ under this fractional linear transformation.

We know that $f(0) = 2i$, it is also easy to see that $f(\infty) = 2$ and $f(i) = 1+i$.

Since fractional linear transformations map lines and circles to lines and circles, and $\{x + iy \in \mathbb{C} \mid x = 0\}$ is a line, its image under f is a line or a circle.

Since $0, \infty, i$ all lie on $\{x + iy \in \mathbb{C} \mid x = 0\}$, the image of this line is a line or circle going through $2i, 2, 1 + i$. It is easy to see that these three points lie in the line $\{x + iy \in \mathbb{C} \mid x + y = 2\}$, so it is the image of $\{x + iy \in \mathbb{C} \mid x = 0\}$ under f.