MATH 54 FINAL

May 12 2016 3-6pm

Your Name Student ID	SOLUTIONS

Please exchange student IDs to record the

names of your two closest seat neighbors	

Do not turn this page until you are instructed to do so.

No material other than simple writing utensils may be used. Show all your work in this exam booklet. There are blank pages in between the problems for scratch work.
If you want something on an extra page to be graded, label it by the problem number and write "XTRA" on the page of the actual problem.
In the event of an emergency or fire alarm leave your exam on your seat and meet with your GSI or professor outside.
If you need to use the restroom, leave your exam with a GSI while out of the room.

Your grade is determined from the following 5 problems, each of which has questions (a), (b), (c).

Each part of (a) yields either full or no credit, but you still have to show your work in calculations.

(b),(c) parts can yield partial credit, in particular for explanations and documentation of your approach, even when you don't complete the calculation. When asked to explain/show/prove, you should make clear and unambiguous statements, using a combination of formulas and words or arrows. (The graders will disregard formulas whose meaning is unclear.) But most importantly ...

... don't panic!

[8] **1(a)** Fill in the ... below.

If $A \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathrm{Nul}(A)$ is spanned by $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then the solution set of $Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in parametric vector form is ... $\underline{x}(t) = \begin{bmatrix} 0 \\ 4 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}^{-1} = \ldots \quad \begin{bmatrix} 1/3 & -1/3 & -2/3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{array}{l} -\text{row } 2 \\ \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & -2 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} -2\text{row } 3 \\ +2\text{row } 3 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & -1 & -2 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & -1/3 & -2/3 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\mathrm{x}) = \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}} \mathrm{x}$

is .... (one-to-one / onto / both / neither).

det = 2

$\Downarrow$

invertible

[6] **1(b)** Determine whether $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ are linearly independent by

- giving the general definition for linear independence of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$,
- translating linear independence into a property of a matrix,
- checking whether that property is true.

$$c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 = \underline{0} \quad \text{only for } c_1 = c_2 = c_3 = 0$$

$$\Updownarrow$$

$$[\underline{v}_1\ \underline{v}_2\ \underline{v}_3]\,\underline{x} = \underline{0} \quad \text{has only solution } \underline{x} = \underline{0}$$

$$\Updownarrow$$

echelon form of $[\underline{v}_1\ \underline{v}_2\ \underline{v}_3]$ has no free variables
(pivot each column)

CHECK:

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -1 & 2 \\ 0 & \boxed{-2} & 2 \\ 0 & 0 & \boxed{2} \end{bmatrix} \implies \underline{\underline{\text{lin. indep.}}}$$

pivot each column

[6] **1(c)** Let $T : \mathbb{R}^2 \to \mathbb{R}^2, \mathbf{x} \mapsto A\mathbf{x}$ be the linear transformation given by the matrix $A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$.

Find a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ of $\mathbb{R}^2$ so that the linear transformation in $\mathcal{B}$-coordinates is $[T]_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

eigenvalues $2 \pm i$

$(2-i)$ eigenvector : $\begin{bmatrix} 3-(2-i) & 1 \\ -2 & 1-(2-i) \end{bmatrix} \underline{z} = \underline{0}$

$$\Longleftrightarrow \begin{bmatrix} 1+i & 1 \\ \underbrace{-2}_{\substack{\parallel \\ (1+i)(-1+i)}} & \underbrace{-1+i}_{\substack{\parallel \\ 1(-1+i)}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longleftrightarrow (1+i) z_1 + z_2 = 0$$

$$\Longleftrightarrow \underline{z} = (\text{complex scalar} \neq 0) \cdot \begin{bmatrix} -1 \\ 1+i \end{bmatrix}$$

By rule from book: $\underline{b}_1 = \mathrm{Re} \begin{bmatrix} -1 \\ 1+i \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\underline{b}_2 = \mathrm{Im} \begin{bmatrix} -1 \\ 1+i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

other solutions from other choices of eigenvectors (ie complex scalars)

are any $\underline{b}_1 \neq \underline{0}$ with $\underline{b}_2 = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \underline{b}_1$

(because we are solving $A\underline{b}_1 = 2\underline{b}_1 + \underline{b}_2$
$A\underline{b}_2 = -\underline{b}_1 + 2\underline{b}_2$)

[8] **2(a)** Fill in the ... below.

The range of a linear transformation $T : V \to W$ is ... all w in W so that $T(x) = w$ has a solution x in V.

alternative: $\{T(v) \mid v \in V\}$

Solutions x of a linear equation $T(x) = b$ are unique if the linear transformation T is ... one-to-one

alternative: "has kernel $= \{0\}$"

A set of vectors $v_1, \ldots, v_p$ in a vector space V is a basis if ... they are linearly independent and span V.

A nonempty subset H of a vector space V is a subspace of V if ... it is closed under

addition $(v, w \in H \Rightarrow v + w \in H)$

and

scaling $(v \in H,\ c \text{ scalar} \Rightarrow cv \in H)$

alternative: $v, w \in H$, c scalar $\Rightarrow$ $cv + w$ in H

[6] **2(b)** Find the matrix that represents the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ given by $T(p) = \frac{\mathrm{d}}{\mathrm{d}t}p$ with respect to the standard basis of $\mathbb{P}_2$.

coordinates

$e_0(t) = 1 \qquad T(e_0) = 0 \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$e_1(t) = t \qquad T(e_1) = 1 = e_0 \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$e_2(t) = t^2 \qquad T(e_2) = 2t = 2e_1 \qquad \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$$\Rightarrow [T]_{\{e_0, e_1, e_2\}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

[6] **2(c)** Let $T : V \to W$ be a linear transformation between vector spaces, and let H be a subspace of V. Show that $T(H) = \{T(v) \mid v \in H\}$ is a subspace of W.

to show: $\underbrace{w_1, w_2 \text{ in } T(H)}$, c scalar $\Rightarrow cw_1 + w_2$ in $T(H)$

$\Updownarrow$

$w_1 = T(v_1)$
$w_2 = T(v_2)$ for some v_1, v_2 in H

$\Downarrow$

$$cw_1 + w_2 = cT(v_1) + T(v_2)$$
$$= T(\underbrace{cv_1 + v_2}_{\text{in } H \text{ since } H \text{ is subspace}}) \quad \text{by linearity of } T$$

$\Downarrow$

$cw_1 + w_2 = T(v)$ for $v = cv_1 + v_2$ in H

$\Downarrow$

$cw_1 + w_2$ in $T(H)$

[8] **3(a)** Fill in the ... below.

The general solution of $y'' + 4y' + 5y = 0$ is $y(t) = \ldots e^{-2t}(a\cos t + b\sin t)$

$$\left(\begin{aligned} & r^2 + 4r + 5 = 0 \\ \Leftrightarrow\ & r = -2 \pm \sqrt{2^2 - 5} \\ & = -2 \pm i \end{aligned} \right)$$

$y'' + 4y' + 5y = 3\cos 2t$ has a particular solution of the form (with undetermined coefficients)

$y(t) = \ldots A\cos 2t + B\sin 2t$

alternative: Re or Im of Ce^{2it} for complex C

The general solution of $(D+3)^4 D[y] = 0$ is

$y(t) = \ldots C_0 + C_1 e^{-3t} + C_2 t e^{-3t} + C_3 t^2 e^{-3t} + C_4 t^3 e^{-3t}$

[6] **3(b)** Find the general solution of $y'' - y = e^t$.

homogeneous eq. general sol. $c_1 e^t + c_2 e^{-t}$

$r^2 - 1 = 0$

$\Leftrightarrow r = \pm 1$

inhomogeneous eq.

Ansatz: $y(t) = a t e^t$ (resonance)

plug in: $y' = a e^t + a t e^t$

$y'' = a e^t + a e^t + a t e^t$

$y'' - y = 2a e^t + a t e^t - a t e^t = 2a e^t$

$\parallel$

e^t

solve: $1 = 2a \Leftrightarrow a = \frac{1}{2}$

general sol.: $y(t) = \frac{1}{2} t e^t + c_1 e^t + c_2 e^{-t}$

[6] **3(c)** Calculate the Wronskian $W(0)$ for the functions $\sin x,\ \sin 2x,\ \cos x$ at $x = 0$ and explain what this says about linear (in)dependence of the functions.

Then give a different argument that proves linear (in)dependence of these functions.

$$W(x) = \det \begin{bmatrix} \sin x & \sin 2x & \cos x \\ \cos x & 2\cos 2x & -\sin x \\ -\sin x & -4\sin 2x & -\cos x \end{bmatrix}$$

$$W(0) = \det \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \underline{\underline{0}} \longrightarrow \text{no conclusion on linear (in) dependence}$$

To prove linear independence:

$$c_1 \sin x + c_2 \sin 2x + c_3 \cos x = 0$$

$x=0$: $\quad c_3 = 0$

$x=\frac{\pi}{2}$: $\quad c_1 = 0$

$x=\frac{\pi}{4}$: $\quad \frac{\sqrt{2}}{2} c_1 + c_2 + \frac{\sqrt{2}}{2} c_3 = 0 \quad \Rightarrow c_2 = 0$

$$\Longleftrightarrow c_1 = c_2 = c_3 = 0$$

alternative: $W(\pi/4) = \ldots \neq 0$

[8] **4(a)** Fill in the ... below.

If A $\begin{bmatrix} 1+3i \\ 2-5i \end{bmatrix} = \sqrt{2}\,i \begin{bmatrix} 1+3i \\ 2-5i \end{bmatrix}$, then the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = \ldots\ C_1 \Big(\underbrace{\cos\sqrt{2}t \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \sin\sqrt{2}t \begin{bmatrix} 3 \\ -5 \end{bmatrix}}_{\text{real part of}} \Big) + C_2 \Big(\underbrace{\cos\sqrt{2}t \begin{bmatrix} 3 \\ -5 \end{bmatrix} + \sin\sqrt{2}t \begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{imaginary part of}} \Big)$$

$$\left(\text{complex solution} \quad e^{\sqrt{2}it} \begin{bmatrix} 1+3i \\ 2-5i \end{bmatrix} = (\cos\sqrt{2}t + i\sin\sqrt{2}t)\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 3 \\ -5 \end{bmatrix} \right) \right)$$

If A has matrix exponential function $e^{tA} = e^{-2t}\begin{bmatrix} 1 & 3t \\ 0 & 1 \end{bmatrix}$ then the solution of $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$ is ...

$$\underline{x}(t) = e^{tA}\underline{x}(0) = e^{-2t}\begin{bmatrix} 1 & 3t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 7 \end{bmatrix} = e^{-2t}\begin{bmatrix} 9+21t \\ 7 \end{bmatrix}$$

$e^{A+B} = e^A e^B$ holds for matrices A, B when ... $AB = BA$

[6] **4(b)** Find the general solution of $\mathbf{x}' = \overbrace{\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}}^{A} \mathbf{x} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ and explain why there cannot be any other solutions, using only definitions and the following information (no theorems etc.):

1.) $L(\mathbf{x}) = \mathbf{x}' - \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \mathbf{x}$ is a linear transformation $L : V \to V$ on the vector space V of smooth functions with values in $\mathbb{R}^2$.

2.) The kernel of L is spanned by $\begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}$ and $\begin{bmatrix} -\cos 2t \\ \sin 2t \end{bmatrix}$.

3.) $\mathbf{x}_p(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ solves $L(\mathbf{x}_p) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

$$\underline{x}' = A\underline{x} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$\Updownarrow \text{ def. of } L$$

$$L(\underline{x}) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$\Updownarrow (3)$$

$$L(\underline{x}) = L(\underline{x}_p)$$

$$\Updownarrow (1)$$

$$L(\underline{x} - \underline{x}_p) = \underline{0}$$

$$\Updownarrow (2)$$

$$\underline{x} - \underline{x}_p \text{ in span} \left\{ \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}, \begin{bmatrix} -\cos 2t \\ \sin 2t \end{bmatrix} \right\}$$

$$\Updownarrow$$

$$\underline{x}(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + C_1 \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix} + C_2 \begin{bmatrix} -\cos 2t \\ \sin 2t \end{bmatrix}$$

("$\Downarrow$" shows all solutions are of this form

"$\Uparrow$" shows all $\underline{x}$ of this form are solutions)

[6] **4(c)** Solve $\mathbf{x}'(t) = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \mathbf{x}(t) + \underbrace{\begin{bmatrix} e^{-t} t^2 \\ e^t t^{-2} \end{bmatrix}}_{\underline{f}}$, $\mathbf{x}(1) = \begin{bmatrix} \sqrt{2} e^{-1} \\ e^2 \end{bmatrix}$ by the following steps:

- A fundamental matrix for $\mathbf{x}'(t) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t)$ is $X(t) = \ldots e^{t\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{bmatrix}$

- The variation of parameters Ansatz is $\mathbf{x}(t) = \ldots e^{tA} \underline{c}(t)$
with an unknown $\mathbb{R}^2$-valued function $\mathbf{c}(t)$.

- Plugging this Ansatz into the inhomogeneous ODE system yields ...

$$\left.\begin{aligned} \underline{x}' &= A e^{tA} \underline{c} + e^{tA} \underline{c}' \\ &\overset{?}{\|} \\ A\underline{x} + \underline{f} &= A e^{tA} \underline{c} + \underline{f} \end{aligned}\right\} \Longleftrightarrow e^{tA} \underline{c}' \overset{?}{=} \underline{f} \Longleftrightarrow \underline{c}' \overset{?}{=} e^{-tA} \underline{f}$$

$$\Longleftrightarrow \underline{c}'(t) \overset{?}{=} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} e^{-t} t^2 \\ e^t t^{-2} \end{bmatrix} = \begin{bmatrix} t^2 \\ t^{-2} \end{bmatrix}$$

- Plugging this Ansatz into the initial condition yields ...

$$\begin{bmatrix} \sqrt{2} e^{-1} \\ e^2 \end{bmatrix} \overset{?}{=} \underline{x}(1) = e^{A} \underline{c}(1) = \begin{bmatrix} e^{-1} & 0 \\ 0 & e \end{bmatrix} \underline{c}(1)$$

$$\Longleftrightarrow \underline{c}(1) = \begin{bmatrix} \sqrt{2} \\ e \end{bmatrix}$$

- Solving these for $\mathbf{c}$ leads to the integral formula

$\mathbf{c}(t) = \mathbf{c}(1) + \int_1^t \mathbf{c}'(s)\, \mathrm{d}s = \ldots$

$$= \begin{bmatrix} \sqrt{2} \\ e \end{bmatrix} + \int_1^t \begin{bmatrix} s^2 \\ s^{-2} \end{bmatrix} ds = \begin{bmatrix} \sqrt{2} + \left[\frac{1}{3} s^3\right]_{s=1}^{s=t} \\ e + \left[-s^{-1}\right]_{s=1}^{s=t} \end{bmatrix} = \begin{bmatrix} \sqrt{2} + \frac{1}{3} t^3 - \frac{1}{3} \\ e - t^{-1} + 1 \end{bmatrix}$$

- The final solution is $\mathbf{x}(t) = \ldots e^{tA} \underline{c}(t) = \begin{bmatrix} e^{-t}\left(\sqrt{2} + \frac{1}{3} t^3 - \frac{1}{3}\right) \\ e^{t}\left(e - t^{-1} + 1\right) \end{bmatrix}$

[8] **5(a)** Fill in the ... below.

The Fourier series of a continuous $2L$-periodic function f is

$$f(x) \sim \ldots \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}x\right) + b_n \sin\left(n\frac{\pi}{L}x\right)$$

with

$$a_n = \ldots \frac{1}{L}\int_0^{2L} f(x)\cos\left(n\frac{\pi}{L}x\right)dx$$

$$b_n = \ldots \frac{1}{L}\int_0^{2L} f(x)\sin\left(n\frac{\pi}{L}x\right)dx$$

$1\cdot\frac{2\pi}{8\pi}x \qquad 2\cdot\frac{2\pi}{8\pi}x$

The coefficients of the 8π-periodic Fourier series of $f(x) = \underbrace{4}_{=b_1}\sin\left(\frac{1}{4}x\right)\underbrace{-7}_{=a_2}\cos\left(\frac{1}{2}x\right) + \cos\left(\frac{\pi}{4} - x\right)$ are

$a_0 = \ldots 0$ $\quad a_1 = \ldots 0$ $\quad a_2 = \ldots -7$ $\quad a_3 = \ldots 0$ $\quad a_4 = \ldots \sqrt{2}/2$ $\quad a_5 = \ldots 0$

$b_1 = \ldots 4$ $\quad b_2 = \ldots 0$ $\quad b_3 = \ldots 0$ $\quad b_4 = \ldots \sqrt{2}/2$ $\quad b_5 = \ldots 0$ $\quad b_6 = \ldots 0$

$$\cos\left(\frac{\pi}{4}-x\right) = Re\left(e^{i(\pi/4 - x)}\right) = Re\left(\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)(\cos x - i\sin x)\right)$$

$$= \underbrace{\frac{\sqrt{2}}{2}}_{a_4}\cos \underbrace{x}_{4\cdot\frac{2\pi}{8\pi}x} + \underbrace{\frac{\sqrt{2}}{2}}_{b_4}\sin x$$

[6] **5(b)** Find the solution to the initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = 5\,\frac{\partial^2 u}{\partial x^2} \qquad \text{for} \quad 0 < x < \pi,\ t > 0,$$

$$u(0,t) = 0, \qquad u(\pi,t) = 0 \qquad \text{for} \quad t > 0,$$

$$u(x,0) = \sum_{n=1}^{\infty} 3^{-n}\sin(nx), \qquad \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} n^{-2}\sin(nx) \qquad \text{for} \quad 0 < x < \pi.$$

Fourier Ansatz: $u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin nx$

PDE:

$$\left.\begin{aligned} \partial_t^2 u &= \sum B_n'' \sin nx \\ &\;\;\| \\ 5\partial_x^2 u &= \sum 5 \cdot B_n (-n^2) \sin nx \end{aligned}\right\} \iff B_n'' = -5n^2 b_n$$

initial conditions:

$$\sum B_n(0) \sin nx = \sum 3^{-n} \sin nx \iff B_n(0) = 3^{-n}$$

$$\sum B_n'(0) \sin nx = \sum n^{-2} \sin nx \iff B_n'(0) = n^{-2}$$

solve: $B_n(t) = a_n \cos\sqrt{5}nt + b_n \sin\sqrt{5}nt$

$$3^{-n} = a_n$$

$$n^{-2} = \sqrt{5}\, b_n \iff b_n = \frac{1}{\sqrt{5}} n^{-2}$$

plug back:

$$u(x,t) = \sum_{n=1}^{\infty} \left(3^{-n} \cos\sqrt{5}nt + \frac{1}{\sqrt{5}} n^{-2} \sin\sqrt{5}nt \right) \sin nx$$

[6] 5(c) Determine ODE's and initial conditions (but do not solve these!) for the coefficient functions $C_0(t), C_1(t), C_2(t), \ldots$ of any solution of the form $u(x,t) = C_0(t) + \sum_{n=1}^{\infty} C_n(t)\cos(nx)$ to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-t}\cos(3x) + (\sin(5t))^2 \qquad \text{for} \quad 0 < x < \pi,\ t > 0,$$

$$u(x,0) = \underbrace{\sqrt{5}}_{n=0} + \underbrace{\sqrt{3}}_{n=1}\cos(x) + \underbrace{\sqrt{2}}_{n=3}\cos 3x \qquad \text{for} \quad 0 < x < \pi.$$

$$\| $$

$$C_0(0) + \sum_{n=1}^{\infty} C_n(0)\cos(nx) \iff C_n(0) = \begin{cases} \sqrt{5} & ; n=0 \\ \sqrt{3} & ; n=1 \\ \sqrt{2} & ; n=3 \\ 0 & ; n=2,4,5,\ldots \end{cases}$$

$$\frac{\partial u}{\partial t} = C_0' + \sum_{n=1}^{\infty} C_n' \cos nx$$

$$\|$$

$$\frac{\partial^2 u}{\partial x^2} + e^{-t}\cos(3x) + \sin^2 5t$$

$$\|$$

$$\sum_{n=1}^{\infty} C_n(-n^2)\cos nx + e^{-t}\cos(\underset{n}{\underset{\|}{3}}x) + \sin^2 5t \cdot \cos 0x$$

$$\Updownarrow$$

$$C_n' = \begin{cases} \sin^2 5t & ; n=0 \\ -9C_n + e^{-t} & ; n=3 \\ -n^2 C_n & ; n=1,2,4,5,\ldots \end{cases}$$