EECS 16A Designing Information Devices and Systems I
Fall 2015 Anant Sahai, Ali Niknejad

Final Exam

Exam location: RSF Fieldhouse, Back Left, last SID 6, 8, 9

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ______________, ______________ ________________
(last) (first) (signature)

PRINT your Unix account login: ee16a-____________

PRINT your discussion section and GSI (the one you attend): ________________________________

Row Number (front row is 1): ________ Seat Number (left most is 1): ________
Name and SID of the person to your left: ________________________________

Name and SID of the person to your right: ________________________________

Name and SID of the person in front of you: ________________________________

Name and SID of the person behind you: ________________________________

Section 0: Pre-exam questions (3 points)

1. **What was your favorite lab/homework-problem in 16A? What did you like best about it? (1 pt)**

2. **Describe how it makes you feel when you work with a TA or friend to understand something? (2 pts)**

Do not turn this page until the proctor tells you to do so. You can work on Section 0 above before time starts.

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[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

Section 1: Straightforward questions (50 points)

Unless told otherwise, you must show work to get credit. There will be very little partial credit given in this section. **You get two drops: do 5 out of the following 7 questions. (We will grade all 7 and keep the best 5 scores.)** *Each problem is worth 10 points. No bonus for getting them all right so skip anything that is taking too much time.*

3. Finding determinant

Compute the determinant of this 3×3 matrix by using Gaussian elimination.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 0 & 4 & 5 \end{bmatrix}$$

4. Back to Basis

Find the matrix that changes the coordinate representation of a vector in the basis $\left\{ \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ to a coordinate representation in the basis $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \end{bmatrix} \right\}$. No need to simplify.

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5. Eigenspaces

Find a basis for the eigenspace corresponding to the eigenvalue $\lambda = 3$ for the following matrix A:

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}$$

6. Show It

Let n be a positive integer. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of k linearly dependent vectors in $\mathbb{R}^n$. Show that for any $(n \times n)$ matrix A, the set $\{A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_k\}$ is a set of linearly dependent vectors.

Your argument should be concise and mathematical.

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7. Linear Recurrence

Suppose the sequence $x_0, x_1, \ldots, x_{t-1}, x_t, x_{t+1}, \ldots$ is defined recursively as follows:

$$\begin{aligned} x_0 &= 0 \\ x_1 &= 1 \\ x_t &= -3x_{t-1} + 4x_{t-2} \end{aligned}$$

So, the first few terms in this sequence are: $0, 1, -3, 13, -51, \ldots$.

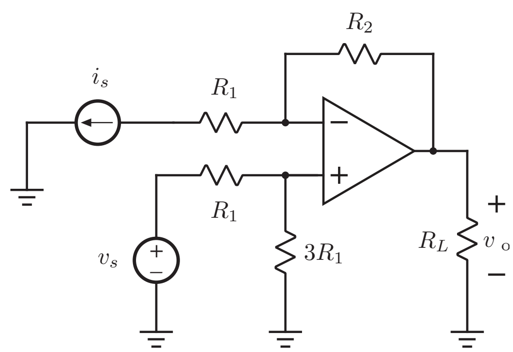
It turns out that by using matrix notation and diagonalization, we can get:

$$\begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} = PD^{t-1}P^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where D is a diagonal matrix. **Find P and D.** (You don't need to invert P.)

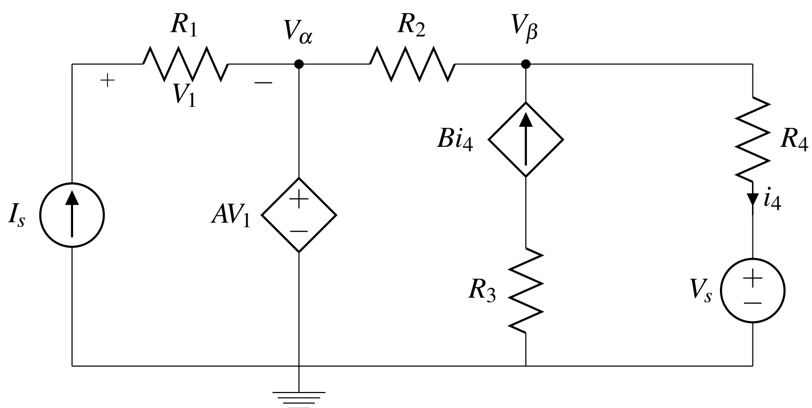
8. Golden Rules

Find v_o, the voltage at the output of the opamp (i.e. across the load R_L) for general resistor and source values.



9. Nodal Analysis

For the following circuit, which of the (a),(b),(c),(d) set of equations is correct (one of them is definitely correct) and sufficient to give us both V_α, V_β assuming that we knew I_s, V_s and all the values for R_1, A, R_2, B, R_3, R_4?



(a)	(b)	(c)	(d)
$V_\alpha = AI_sR_1$	$V_\alpha = AI_sR_1$	$V_\alpha = AI_sR_1$	$V_\alpha = AI_sR_1$
$V_\beta = \frac{(B+1)V_sR_2 + AI_sR_1R_4}{(B+1)R_2 + R_4}$	$V_\beta = -100V$	$V_\beta = \frac{(B-1)V_sR_2 - AI_sR_1R_4}{(B-1)R_2 - R_4}$	$V_\beta = \frac{(B+1)V_sR_2 - AI_sR_1R_4}{(B+1)R_2 - R_4}$

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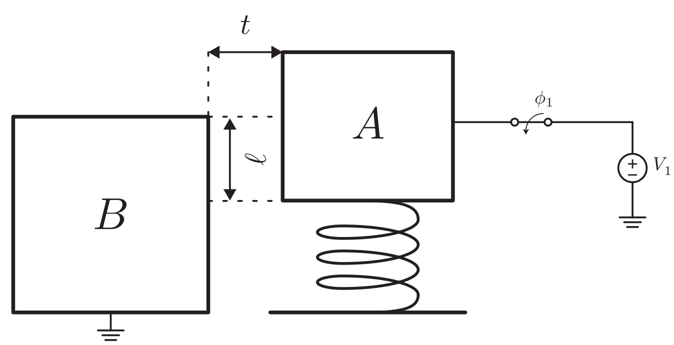
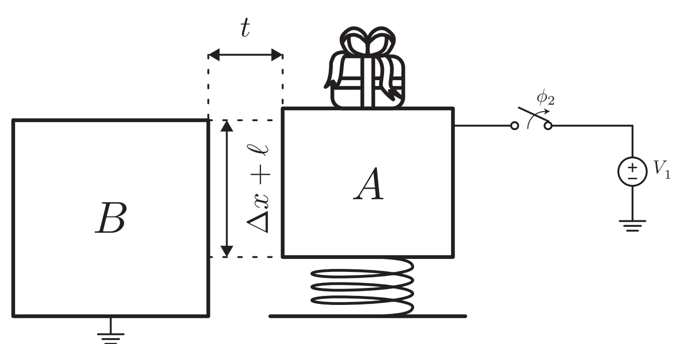
Section 2: Free-form Problems (101+15 points)

10. Airport (41+10 points)

In this question you will design a scale (illustrated below) for an airport check-in counter. A bag is placed on the metal platform A and will move the platform down with respect to the fixed plate B on the left. The vertical displacement, Δx, is proportional to the weight of the bag. A parallel plate capacitor is formed by the metal platform A and the fixed metal plate B. The fixed metal plate B is connected to ground. Assume the separation t and the width w (into the page) stay constant along the entire platform and plate. (You do not need the specific value for t or w to solve any part of this problem.) When there is no bag on the platform, let's call the capacitance C_{min}.





(a) (5 points) In the first phase ϕ_1, *before placing the bag on the platform*, the platform is charged to a voltage $V_1 = 5V$. **Calculate the amount of charge Q_1 on the capacitor in terms of C_{min}.** Just tell us the charge on the platform A.

(b) (8 points) **Calculate the capacitance between the platform and the plate when the platform is displaced by a positive distance** Δx. This displacement makes the effective plates of the capacitor bigger. Assume the bag is non-conductive plastic. Assume parallel plate capacitance (ignore fringing fields). Express your answer in terms of only C_{min}, ℓ, and Δx.

(c) (5 points) In the second phase ϕ_2, the switch is opened (disconnecting the voltage source) and then the bag is placed on the platform A. **Calculate the second phase voltage** V_2 **across the capacitor in terms of only** Δx, ℓ, **and the first phase voltage** $V_1 = 5V$.

(Hint: As you increase Δx, the voltage V_2 should decrease.)

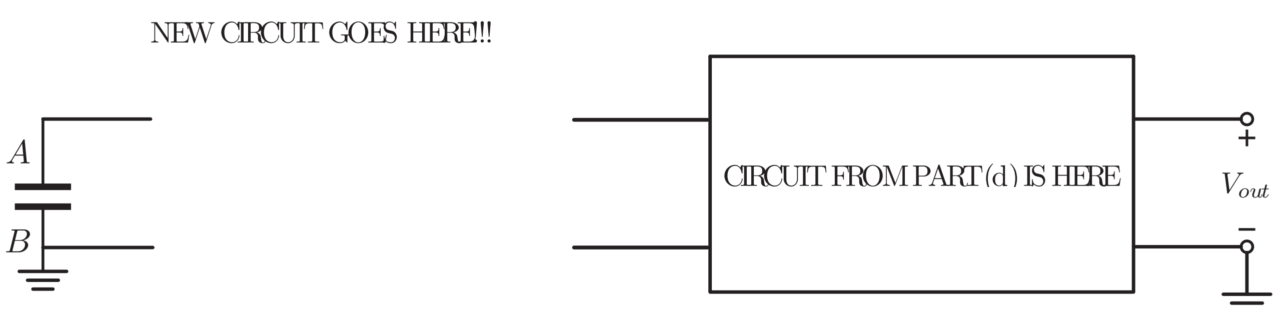
(d) (10 points) We decide that we want to amplify the voltage on the capacitor by a factor of -2. Assuming (for now) that the capacitor acts like an ideal voltage source whose value is V_2 Volts (as shown below), **please draw a circuit that could perform this task.** You are allowed to use wires, resistors, and a single golden-rule op-amp. Your circuit should conceptually fit in the box below.

YOUR CIRCUIT GOES HERE!!

(e) (5 points) Now, instead of an ideal voltage source, we want to connect the circuit from the previous part to the capacitor in the bag weighing setup. It turns out that the circuit in the previous part would load (i.e. draw current to/from) the capacitor if we attached it. **Explain why this happens.**

(Hint: explain what is the equivalent resistance that the capacitor sees to ground when connected to the circuit in the previous part. This might require a different approach than the black-box approach to calculating Thevenin equivalence.)

(f) (8 points) **Draw a circuit that you could place between the capacitor and the circuit from part (d) that would prevent the effects of the loading.** You are allowed to use wires, resistors and one golden-rule op-amp for your new circuit. You don't have to redraw the circuit from part (d). The box below is just to remind you where it would go.



(g) (BONUS 10 points) One last thing. We would like to turn on a light bulb if the bag were greater than 50 lbs. When $\Delta x = \ell$, the weight of the bag is exactly 50 lbs.

Attached to the output of your previous circuits (the whole setup drawn in (f)), **please draw a new circuit that turns a light on if $\Delta x > \ell$ and turns the light off if $\Delta x < \ell$.** Assume the light bulb acts like a 100Ω resistance and it turns on if a nonzero current flows through it. In addition to the lightbulb, you are allowed to use wires, resistors, and a single op-amp in your design for this part. The supply voltages available to you are $10V$ and $-10V$ and you can assume that the op-amp is connected to these as its supply rails. You don't have to redraw what was in (f) – you can start with V_{out} being available to you.

11. Channel Equalization (33 points)

In HW14 and lecture, we have talked about transmitting information through a wireless channel with echos. You don't have to remember the HW problem to solve this question, all relevant information is self-contained here.

If we periodically transmit an n-long vector $\vec{x}$ through a channel with circulant echo matrix H, we receive the n-long vector $\vec{y} = H\vec{x}$.

Helpful Background: In OFDM, we transmit information on the eigenvectors of H. We let

$$\vec{s} = \sum_{\ell=1}^{n} \alpha_\ell \vec{u}_\ell$$

where $\{\alpha_\ell\}$ is the set of messages we are sending and $\vec{u}_\ell$ are eigenvectors of H. If we then choose $\vec{x} = \vec{s}$ itself, then we will receive the n-long vector

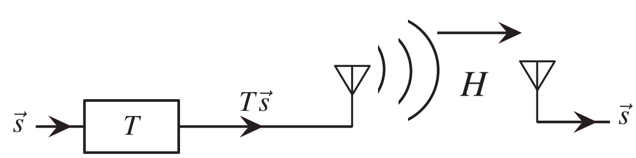
$$\vec{y} = H\vec{x} = \sum_{\ell=1}^{n} \lambda_\ell \alpha_\ell \vec{u}_\ell$$

where $\{\lambda_\ell\}$ is the sequence of eigenvalues of H. Then, if the receiver knows all the λ_ℓ, it can solve for α_ℓ, as long as none of the λ_ℓ are zero.
By changing coordinates to the orthonormal eigenbasis U (consisting of the $\vec{u}_\ell$), we can write

$$U^*\vec{y} = \Lambda\vec{\alpha}$$

where Λ is a diagonal matrix and $U^*\vec{y}$ is the received vector after a change of basis.
One of your homework problems mentioned how pilot tones can be used to estimate Λ. However, a very simple device — for example, a wireless decoder in your clothes — might not be powerful enough to estimate H and divide by the λ_ℓ's. Our goal is to make our device as simple as possible, so to do that we'll make the transmitter (which is plugged into the wall) do all the hard work.

(a) (5 points) We want the transmitter to apply a transformation T to the n-length signal we want to communicate, $\vec{s}$, such that after the transmitted signal $\vec{x} = T\vec{s}$ goes through the channel with circulant echo matrix H, the receiver sees $\vec{y} = \vec{s}$ at its antenna. **Assuming that T exists, what is this transformation T?**



(b) (8 points) **Assuming that T exists, what are the eigenvectors of T, and what are the eigenvalues for T in terms of the eigenvalues of H?**

(Hint: think about diagonalizing H and looking in that basis. Also recall $(AB)^{-1} = (B^{-1}A^{-1})$ if these are square matrices.)

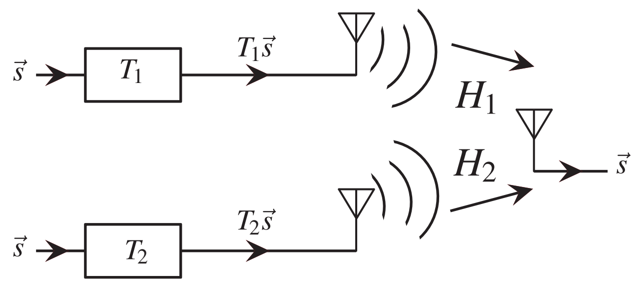
(c) (5 points) **Give a condition on λ_ℓ (an eigenvalue of H) such that if even one of the λ_ℓ satisfies this condition, then T cannot exist and it is impossible to get $\vec{y} = \vec{s}$ in general.**

(This means that at least some of the information in $\vec{s}$ is doomed to be lost forever.)

(d) (15 points) We decide to add a second antenna to our transmitter. These two antennas will broadcast $T_1\vec{s}$ and $T_2\vec{s}$. Suppose $n = 3$ in this part.

The antennas are far apart and so their echo patterns are different. Antenna 1 has circulant echo matrix H_1 with first column $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ and antenna 2 has circulant echo matrix H_2 with first column $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

The receiver receives the 3-long $\vec{y} = H_1T_1\vec{s} + H_2T_2\vec{s}$ which is the sum of the two transmitted signals with their respective echoes.



We can diagonalize H_1 as follows

$$H_1 = \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{3}} & e^{i\frac{4\pi}{3}} \\ 1 & e^{i\frac{4\pi}{3}} & e^{i\frac{8\pi}{3}} \end{bmatrix} \right) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{3}} & e^{i\frac{4\pi}{3}} \\ 1 & e^{i\frac{4\pi}{3}} & e^{i\frac{8\pi}{3}} \end{bmatrix} \right)^{-1}.$$

- **First, write out H_2 and diagonalize it.**
- **And then, give matrices T_1 and T_2 so that $\vec{y} = \vec{s}$.**

It suffices to give T_1 in the eigenbasis for H_1 and T_2 in the eigenbasis for H_2. There are multiple right answers here but we encourage you to just pick whichever you consider the simplest.

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12. One Magical Procedure (27+5 points)

Suppose we have a vector $\vec{x} \in \mathbb{R}^5$ and an $n \times 5$ measurement matrix M defined by column vectors $\vec{c}_1, \cdots, \vec{c}_5$ such that:

$$M\vec{x} = \begin{bmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_5 \\ | & & | \end{bmatrix} \vec{x} \approx \vec{b}$$

We can treat the vector $\vec{b} \in \mathbb{R}^n$ as a noisy measurement of the vector $\vec{x}$, with measurement matrix M and some additional noise in it as well.

You also know that the **true $\vec{x}$ is sparse — it only has two non-zero entries and all the rest of the entries are zero in reality.** Our goal is to recover this original $\vec{x}$ as best we can.

However, your intern has managed to lose not only the measurements $\vec{b}$ but the entire measurement matrix M as well!

Fortunately, you have found a backup in which you have all the pairwise inner-products $\langle \vec{c}_i, \vec{c}_j \rangle$ between the columns of M and each other, as well as all the inner products $\left\langle \vec{c}_i, \vec{b} \right\rangle$ between the columns of M and the vector $\vec{b}$. Finally, you also find the inner-product $\left\langle \vec{b}, \vec{b} \right\rangle$ of $\vec{b}$ with itself.

All the information you have is captured in the following table of inner-products. (These are not the vectors themselves.)

$\langle .,. \rangle$	$\vec{c}_1$	$\vec{c}_2$	$\vec{c}_3$	$\vec{c}_4$	$\vec{c}_5$	$\vec{b}$
$\vec{c}_1$	2	0	1	-1	1	1
$\vec{c}_2$		2	1	-1	-1	-5
$\vec{c}_3$			2	0	-1	2
$\vec{c}_4$				2	-1	6
$\vec{c}_5$					2	-1
$\vec{b}$						29

(So, for example, if you read this table, you will see that the inner product $\langle \vec{c}_2, \vec{c}_3 \rangle = 1$, the inner product $\left\langle \vec{c}_3, \vec{b} \right\rangle = 2$, and that the inner product $\left\langle \vec{b}, \vec{b} \right\rangle = 29$. By symmetry of the real inner product, $\langle \vec{c}_3, \vec{c}_2 \rangle = 1$ as well.)

Your goal is to find which entries of $\vec{x}$ are non-zero, and what their values are.

(a) (4 points) **Use the information in the table above to answer which of the $\vec{c}_1, \ldots, \vec{c}_5$ has the largest magnitude inner product with $\vec{b}$?**

(b) (5 points) Let the vector with the largest magnitude inner product with $\vec{b}$ be $\vec{c}_a$. Let $\vec{b}_p$ be the projection of $\vec{b}$ onto $\vec{c}_a$. **Write $\vec{b}_p$ symbolically as an expression only involving $\vec{c}_a$, $\vec{b}$ and their inner-products with themselves and each other.**

(c) (10 points) **Use the information in the table above to find which of the column vectors $\vec{c}_1, \dots, \vec{c}_5$ has the largest magnitude inner product with the residue $\vec{b} - \vec{b}_p$.**

(Hint: the linearity of inner products might prove useful.)

(d) (8 points) Suppose the vectors we found in parts (a) and (c) are $\vec{c}_a$ and $\vec{c}_c$. These correspond to the components of $\vec{x}$ that are non-zero, that is, $\vec{b} \approx x_a\vec{c}_a + x_c\vec{c}_c$. However, there might be noise in the measurements $\vec{b}$, so we want to find the linear-least-squares estimates $\widehat{x}_a$ and $\widehat{x}_c$. **Write a matrix expression for $\begin{bmatrix}\widehat{x}_a \\ \widehat{x}_c\end{bmatrix}$ in terms of appropriate matrices filled with the inner products of $\vec{c}_a$, $\vec{c}_c$, $\vec{b}$.**

(e) (BONUS: 5 points) **Compute the numerical values of $\widehat{x}_a$ and $\widehat{x}_c$ using the information in the table.**

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Copy of table for convenience:

$\langle .,. \rangle$	$\vec{c}_1$	$\vec{c}_2$	$\vec{c}_3$	$\vec{c}_4$	$\vec{c}_5$	$\vec{b}$
$\vec{c}_1$	2	0	1	-1	1	1
$\vec{c}_2$		2	1	-1	-1	-5
$\vec{c}_3$			2	0	-1	2
$\vec{c}_4$				2	-1	6
$\vec{c}_5$					2	-1
$\vec{b}$						29

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[Doodle page! Draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed. Or tell us a joke or a story.]