In the observer frame the laser light is doppler shifted via relativistic doppler effect:

$$\lambda_{obs} = \sqrt{\frac{1+\beta}{1-\beta}}\,\lambda_{source} \qquad + \quad \lambda = \frac{c}{f} \quad \left(\lambda_{source} = \frac{c}{f_0},\ \lambda_{obs} = \frac{c}{f}\right)$$

Now single slit diffraction: minima when $D\sin\theta = m\lambda$

Small angle means then   first minima: $D\sin\theta = \lambda$

$\sin\theta \simeq \tan\theta \simeq \theta \quad \Rightarrow \quad D\sin\theta = \lambda$ becomes

but here we have $D\tan\theta = \lambda$

$\tan\theta = \frac{y}{L}$ so: $y = \frac{\lambda L}{D}$ So:

i.e. $y_A = \frac{\lambda_{source} L}{D}$, $y_0 = \frac{\lambda_{obs} L}{D}$ $\Delta y = \frac{L}{D}\left(\lambda_{obs} - \lambda_{source}\right)$

So $\Delta y = \frac{Lc}{Df_0}\left(\sqrt{\frac{1+\beta}{1-\beta}} - 1\right)$ $+$ $\Delta y > 0$ means:

Now we solve for v:

$\sqrt{\frac{1+\beta}{1-\beta}} - 1 > 0 \rightarrow \frac{1+\beta}{1-\beta} > 1$

$$\left(\frac{\Delta y D f_0}{Lc} + 1\right)^2 = \frac{1+\beta}{1-\beta}$$

$\hookrightarrow 1+\beta > 1-\beta \rightarrow 2\beta > 0$

call this $\alpha = \frac{\Delta y D f_0}{Lc} + 1$

so $\beta > 0$ means

$\rightarrow (1-\beta)\alpha^2 = 1+\beta$

$v > 0$ and laser is

$\hookrightarrow \beta(1+\alpha^2) = \alpha^2 - 1$

moving toward the slit

$\beta = \frac{\alpha^2 - 1}{1+\alpha^2}$ $\Rightarrow v = c\left[\dfrac{\left(\frac{\Delta y D f_0}{Lc}+1\right)^2 - 1}{1 + \left(\frac{\Delta y D f_0}{Lc}+1\right)^2}\right]$

(to the right)

$v = \beta c$

Problem 2

The frame S' cannot be reached from frame S by a single boost.

We can see that the height h of the triangle has not changed, limiting the boost to the x direction. We see also that the orientation of the triangle has changed— in addition to length contraction in the x direction, the triangle appears to have been reflected through the y axis. We can model this reflection by saying that a hypothetical boost linking the two frames acts on the lower leg of the triangle like a length contraction $L \rightarrow -L/2$ (the negative length represents the change in orientation). Using our length contraction formula, we would conclude that $\gamma < 0$ (implying imaginary velocity), which we know to be impossible. Hence there is no frame S' moving with respect to S in the x direction that would see the triangle as drawn in the second figure. This conclusion, with an equivalent justification, was worth full credit.

Extra Information

In the full theory of relativity, it's actually possible for such a frame to exist. We could, for example, model the orientation flip as a rotation through $180°$ around the y axis. A fact from upper division relativity is that the composition of two boosts along different axes can involve a rotation (among other things, this causes the Thomas precession of electron spin in an atom). Hence there is actually a Lorentz transformation that would link the two frames.

Furthermore, reflections are also allowed in full relativity— we distinguish between "proper" Lorentz transformations (which do not involve orientation changes, i.e. the sign of the volume of a cube is preserved), and "improper" Lorentz transformations (which do involve orientation changes). Thus there is also an improper Lorentz transformation connecting the two frames.

Both of these points are of course beyond the scope of this course, and understanding of the mechanisms was not required for full credit.

Midterm Exam 2 Problem 3 Solution
Spring 2016 PHYS 7C

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Contents

1   Comprehension

Part of the difficulty in this question is comprehension. My suggestion is you should *read through the first paragraph* (before part (a)) *again and again until you understand the situation* and paraphrase it in your own terms. For example in this case, we can *draw the spacetime diagram* and *identifies the events* on it. And then we can write down the information in terms of the *spacetime coordinate of the events*.

   Using the terminology suggested in the exam question, B stands for the back of the spacecraft, F stands for the front of the spacecraft. (O stands for the origin as usual).

1.1 Spacetime Diagram

Spacetime Diagram with $v = 0.28, \gamma = 1.0416$
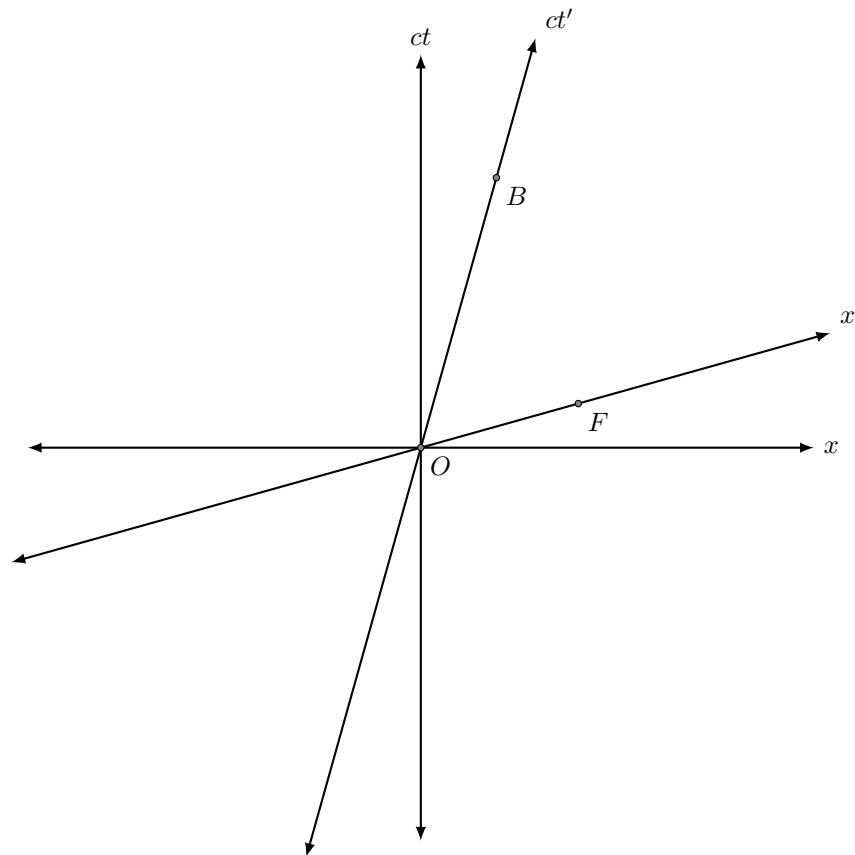
1.2 Information

The following merely paraphrase the given information in the first paragraph in terms of equations:

1.2.1 Origin Set by the Spaceship

According to the diagram given on the exam, the back of spaceship is the origin of S', i.e. $x' = 0$.

1.2.2 Event 1: particle entering the spaceship

In S' frame, $x'^{\mu}_F = (0, L)$ [1]

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[1] The notation x^{μ} denotes the spacetime 4-vector with components (ct, x)

1.2.3 Event 2: particle exiting the spaceship

In S frame, $x_B^\mu = (ct_0, x_0)$
  In S' frame, $x_B'^\mu = (ct_0', 0)$, where t_0' is an unknown

2 Solution

In the following solution, the easiest method is presented first, and then an alternative method *using Lorentz transformation only* is presented.

2.1 Part (a)

On the Spacetime Diagram (section 1.1) above, focus on the path OB, which is the back of the spaceship. One can immediately write down:

$$\boxed{v = \frac{x_0}{t_0}} \tag{1}$$

2.1.1 Part (a) Alternative Method

$$x_B'^\mu = \Lambda(\beta)x_B^\mu, \quad \Lambda(\beta) = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \tag{2}$$

Using the known $x_B'^\mu$ [2] and x_B^μ above, one can calculate the unknown β, hence v.

2.2 Part (b)

We are now focusing on the *path FB*—the particle entered at F and exits at B.
  In S' frame, using the known $x_B'^\mu$ and $x_F'^\mu$, one can find the distance travelled and time taken. Hence

$$u' = -\frac{L}{t_0'}.$$

Realizing t_0' is the *proper time of the spaceship*: when the particle took time t_0' to go from F to B, the back of the spacetime took time t_0' to go from O to B. i.e.

$$t_0 = \gamma t_0'.$$

Final answer:

$$u' = -\frac{\gamma L}{t_0}, \quad \gamma = \frac{1}{\sqrt{1 - \left(\frac{x_0}{ct_0}\right)^2}}$$

Simplified:

$$\boxed{u' = -\frac{cL}{\sqrt{(ct_0)^2 - x_0^2}}} \tag{3}$$

---

[2] only the 2nd coordinate is known, and is enough

2.2.1 Part (b) Alternative Method

To calculate t'_0, one can use equation (2), from the known v, the last unknown t'_0 can be calculated.

2.3 Part (c)

Using the *relativistic velocity addition rule*: $u = \frac{v+u'}{1+\frac{vu'}{c^2}}$, answers from part (a) and (b) can be combined to find the answer to part (c).

Final answer:

$$u = \frac{x_0 - \gamma L}{t_0 - \frac{\gamma x_0 L}{c^2 t_0}}, \quad \gamma = \frac{1}{\sqrt{1 - \left(\frac{x_0}{ct_0}\right)^2}}$$

Simplified:

$$\boxed{\frac{x_0\sqrt{(ct_0)^2 - x_0^2} - ct_0 L}{ct_0\sqrt{(ct_0)^2 - x_0^2} - x_0 L}} \tag{4}$$

2.3.1 Part (c) Alternative Method

One can use the *inverse Lorentz transformation*:

$$x_F^\mu = \Lambda(-\beta)x_F'^\mu, \quad \Lambda(-\beta) = \gamma \begin{pmatrix} 1 & +\beta \\ +\beta & 1 \end{pmatrix} \tag{5}$$

Since $x_F'^\mu, v$ are known, x_F^μ can be solved.

Using the just calculated x_F^μ and the given x_B^μ above, one can calculate the speed by the change of distance over time.

3 Grading Scheme and Common Mistakes

Part (b) seems to be most difficult, so it carries more points than the other:

Part	Grade Points
a	5
b	10
c	5

Holistic grading is then used for each part at a 5 points scales (i.e. part(b) will be $0, 2, 4, 6, 8, 10$).

Generally, I interpret holistic grading as follows:

1. Nice Try

2. Wrong direction

3. Right direction, major mistakes (major algebra, sign, or approximation errors that leads to totally unexpected results)

4. Right direction, minor mistakes (minor algebra, sign, or approximation errors that do not leads to completely wrong qualitative behaviors)

5. Perfect/Almost Perfect

3.1 Comments on Part (a)

Most of the students get this correctly. The main problem is *many of you did not explain how you get the answer.* By mere dimensional analysis, distance over time is velocity, so it would not be very difficult to guess (1) without understanding it. However, I *gave you the benefit of the doubt* here. But in the future I hope you can explain how you come up with the answer, for example, see Part (a) (section 2.1).

3.2 Comments on Part (b)

- Wrong direction: Either one of the following:

  1. fail to identify/represent the 2 events that represents the moving particle to find the speed it has

  2. use a wrong Lorentz transformation/other formula to calculate the velocity from the 2 events

  3. fail to explain how you write down the (wrong) equations at the very beginning

- Major mistake:

  – fail to express the unknown time (e.g. t_0') in terms of the known t_0

The last point in "Wrong direction" is reiterating the fact you need to explain your work. I am usually generous to give you full mark if you got the right answer. However, *if you got the wrong answer*, without you writing down the reasoning in the intermediate steps, it will be *very difficult to distinguish your work from "wrong direction" to "right direction"*, because *you did not show your direction.* In the case of no direction (no explanation/steps for your work), wrong direction is assumed.

Note that a *sign mistake here is regarded as "minor mistake"*, but not "almost perfect". One might claim despite it is not written down, it is obvious because the particle enters from the front to the back. But if it is obvious then it is even more unforgivable to have missed it. i.e. you really have to write down what you know, if you indeed know it. The written answer should be final. I cannot emphasize more the fact that the presentation and organization of your answer is very important.

3.3 Comments on Part (c)

- Nice Try:

  – fail to apply the relativistic velocity additional rule

- Wrong direction:

– correctly quote the relativistic velocity addition rule formula, but fail to concretely apply it, including not able to identify the 2 velocities you are adding up

• Major mistakes:

– correctly quote the relativistic velocity addition rule, but quote a wrong result from part (b). It is not a double penalty in this case, since the answer to part (c) can be calculated independently from part (b), but it is your choice to trust your wrong answer from part (b). See Part (c) Alternative Method (subsection 2.3.1) for explanation for details before the urge to submit regrade request.

3.4   General Comments

This question does not require the use of Lorentz transformation to obtain the answers [3]. *Only time dilation and relativistic velocity addition rule are needed*, together with some basics (e.g. velocity is distance over time). So it is definitely the easier kind of problems.

One difficulty, explained in the beginning in Comprehension (chapter 1), is comprehension and explained there. These comprehension skills are also commonly very important in solving Physics problems in general.

Most of you did not do well in this problem, *having a mean of* 7.8 *and a standard deviation of* 4.86. Some do get full marks though. I guess the main problem is comprehension, since the Physics involved is really easy as explained above. However, this *cannot be an excuse to file a regrade request claiming you "only misunderstand the problem"* and in terms of what you understand, you calculated it correctly (this kind of regrade request do appears in real life). In a certain sense, *many things in Physics are really straight forward if all the abstractness is comprehended.*

Lastly, a word on simplifying answers. As you seen from above, the complicated answers in part (b) and (c), asked to express in terms of the given unknown quality, *can be simplified greatly* (especially part (b)). When I was in Hong Kong, answer not simplified is regarded as wrong answer. Although I grade them very leniently in the exam, I hope to remind you here that simplifying answer is very important. In a certain sense simplifying answers is the most critical effort to solve any problems. Otherwise most problems can be put in this way: given this and that fundamental laws/equations/formula, using this and that initial conditions/information, mathematically there is an answer by solving the systems of equations. . .

In summary, it will be useful in problem solving, especially in exam, to:

1. *paraphrase* the given information e.g. in terms of graphs/diagrams/plots, equations

2. not only write down equations but *explain why/how* you come up with it (e.g. concerning the path OB on the spacetime diagram. . .)

3. express the answer *in terms of the known*, given quantity

4. *simplify* your answer

5. *double check* the sign, units, and dimensions of your final answer

---

[3]Although in each part, an alternative method is given to solve it purely by Lorentz transformation.

$$\begin{pmatrix} E_m \\ pc \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} Mc^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E_+ = p_+ c \\ p_+ \cos\theta \\ p_+ \sin\theta \\ 0 \end{pmatrix} + \begin{pmatrix} E_- = p_- c \\ p_- \cos\theta \\ -p_- \sin\theta \\ 0 \end{pmatrix}$$

$$p_+ \sin\theta = p_- \sin\theta$$
$$p_+ = p_-$$
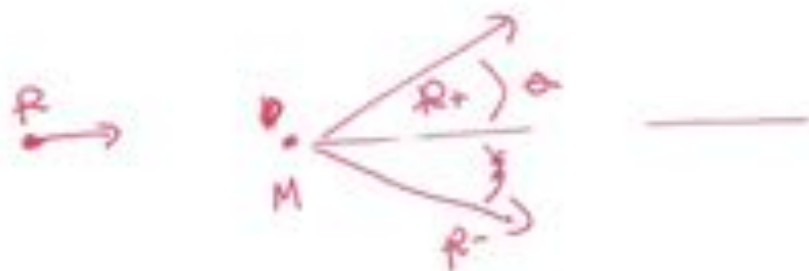
$$(Mc^2)^2 = E_m^2 - p^2 c^2 = (p_+ c + p_- c - Mc^2)^2 - (p_+ \cos\theta + p_- \cos\theta)^2$$

$$M^2 c^4 = (2p_+ c - Mc^2)^2 - (2p_+ \cos\theta)^2 c^2$$

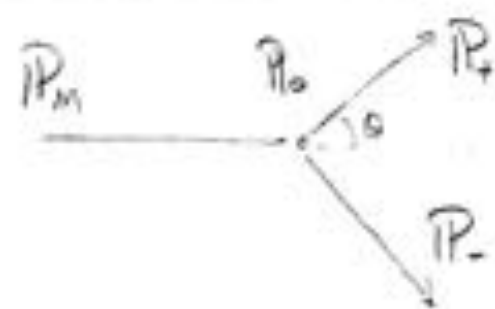$$M^2 c^4 = 4p_+^2 c^2 - 2p_+ Mc^3 + M^2 c^4 - 4p_+^2 c^2 \cos^2\theta$$

$$0 = 4p_+^2 c^2 (1 - \cos^2\theta) - 2p_+ Mc^3$$

$$0 = p_+ c^2 (p_+ \sin^2\theta - Mc)$$

$$p_+ = 0 \qquad \boxed{p_+ = \frac{cM}{\sin^2\theta}}$$

Collision Problem

$$\mathbb{P}_M = (E_M, \vec{p})$$
$$\mathbb{P}_0 = (Mc^2, \vec{0})$$
$$\mathbb{P}_+ = (E_+, \vec{p}_+c)$$
$$\mathbb{P}_- = (E_-, \vec{p}_-c)$$

$$\mathbb{P}_M + \mathbb{P}_0 = \mathbb{P}_+ + \mathbb{P}_-$$

$$\mathbb{P}_- = \mathbb{P}_M + \mathbb{P}_0 - \mathbb{P}_+$$

$$\mathbb{P}_-^2 = (\mathbb{P}_M + \mathbb{P}_0 - \mathbb{P}_+)^T \eta (\mathbb{P}_M + \mathbb{P}_0 - \mathbb{P}_+)$$
$$= \mathbb{P}_M^2 + 2\mathbb{P}_M^T \eta \, \mathbb{P}_0 - 2\mathbb{P}_M^T \eta \, \mathbb{P}_+ + \mathbb{P}_0^2 - 2\mathbb{P}_0^T \eta \mathbb{P}_+ + \mathbb{P}_+^2$$

$$\mathbb{P}_-^2 = m^2 c^4, \quad \mathbb{P}_+^2 = m_+^2 c^4, \quad \mathbb{P}_M^2 = \mathbb{P}_0^2 = M^2 c^4$$

$$m^2 c^4 = 2M^2 c^4 + m_+^2 c^4 + 2[E_M Mc^2 - 0] - 2[E_M E_+ - \vec{p} \cdot \vec{p}_+ c^2]$$
$$- 2[Mc^2 E_+ - 0]$$

$$0 = M^2 c^4 + E_M Mc^2 - E_M E_+ + p p_+ c^2 \cos\theta - E_+ Mc^2$$

$$E_+ \cong p_+ c$$

$$0 = M^2 c^4 + E_M Mc^2 - \{E_M p_+ c - pc \, p_+ c \cos\theta + Mc^2 p_+ c\}$$

$$0 = M^2 c^3 + E_M Mc - [E_M - pc\cos\theta + Mc^2] p_+$$

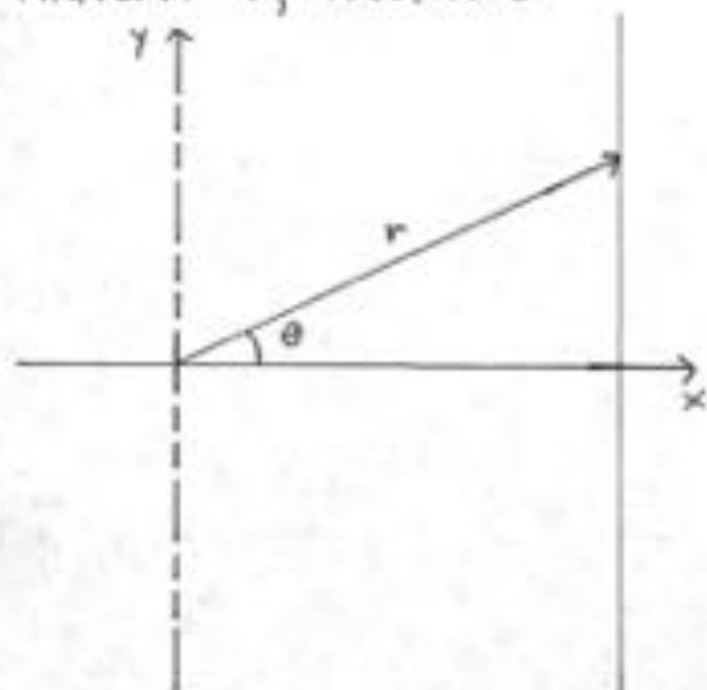$$\boxed{p_+ = \frac{Mc[Mc^2 + E_M]}{E_M + Mc^2 - pc\cos\theta}}$$
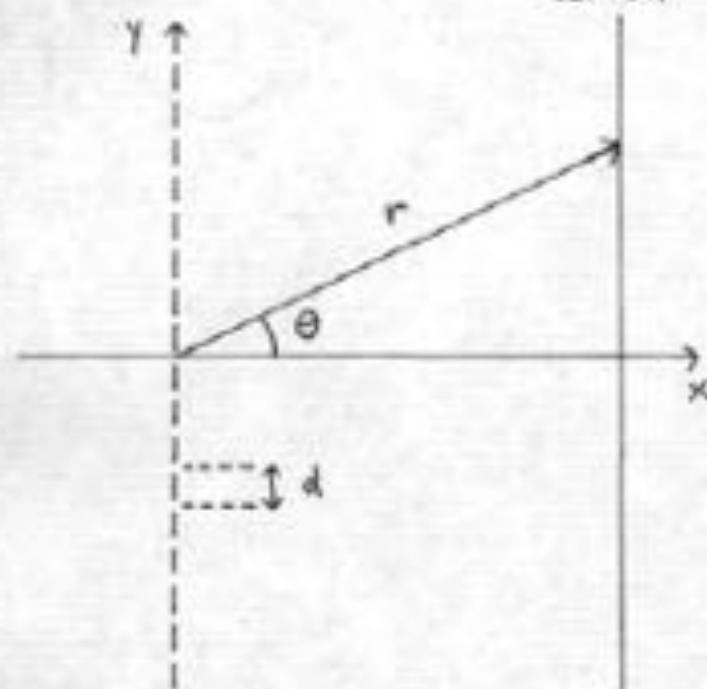
Figure B

screen

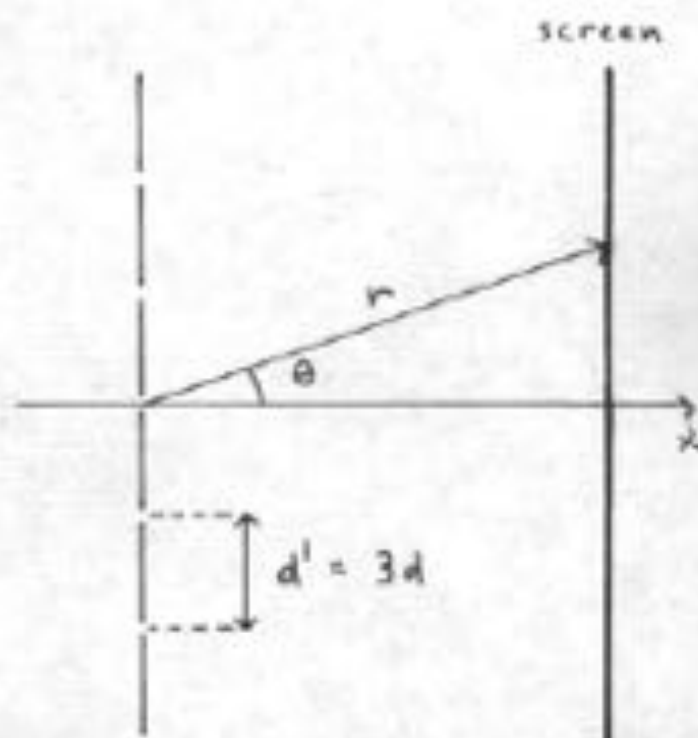Figure A : diffraction grating:
# of slits = 6N+3
slit separation = d

Figure C : diffraction grating:
# of slits = $\frac{1}{3} \cdot (6N+3) = 2N+1$
slit separation = d' = 3d

By the superposition principle for electric fields, the electric field at the screen for the configuration in Figure B is equal to

$$\vec{E}_B = \vec{E}_A - \vec{E}_c$$

$$E_{By} = E_{Ay} - E_{cy}$$

The electric field at the screen for the diffraction grating in Figure A is given in the problem statement to be

$$E_{Ay} = E_o \frac{\sin\left(\beta\left(3N + \frac{3}{2}\right)\right)}{\sin\left(\frac{\beta}{2}\right)} \sin(kr - wt) \qquad \text{where } \beta = kd\sin\theta$$

Similarly, the electric field at the screen for the diffraction grating in Figure C is

$$E_{cy} = E_o \frac{\sin\left(\beta'\left(N + \frac{1}{2}\right)\right)}{\sin\left(\frac{\beta'}{2}\right)} \sin(kr - wt) \qquad \text{where } \beta' = kd'\sin\theta = 3kd\sin\theta$$
$$= 3\beta$$

$$= E_o \frac{\sin\left(\beta\left(3N + \frac{3}{2}\right)\right)}{\sin\left(\frac{3\beta}{2}\right)} \sin(kr - wt)$$

Therefore, the electric field E_{By} is

$$E_{By} = E_{Ay} - E_{cy}$$
$$= E_o \left[\frac{1}{\sin\left(\frac{\beta}{2}\right)} - \frac{1}{\sin\left(\frac{3\beta}{2}\right)}\right] \sin\left(\beta\left(3N + \frac{3}{2}\right)\right) \sin(kr - wt)$$

The intensity (irradiance) at the screen for the configuration in Figure B is

$$I = \epsilon_o c \langle \vec{E}_B^2 \rangle = \epsilon_o c \langle E_{By}^2 \rangle$$

$$= \underbrace{\epsilon_o c E_o^2}_{= I_o} \left[\frac{1}{\sin\left(\frac{\beta}{2}\right)} - \frac{1}{\sin\left(\frac{3\beta}{2}\right)}\right]^2 \sin^2\left(\beta\left(3N + \frac{3}{2}\right)\right) \underbrace{\langle \sin^2(kr - wt)\rangle}_{= \frac{1}{2}}$$

$$= \frac{I_o}{2}\left[\frac{1}{\sin\left(\frac{\beta}{2}\right)} - \frac{1}{\sin\left(\frac{3\beta}{2}\right)}\right]^2 \sin^2\left(\beta\left(3N + \frac{3}{2}\right)\right)$$

If we simplify the term in square brackets, we can rewrite this as

$$I = 2I_o \left[\frac{\cos\beta}{\sin\left(\frac{3\beta}{2}\right)} \sin\left(\beta\left(3N + \frac{3}{2}\right)\right)\right]^2$$