Math 104, Final Examination

Instructor: Guoliang Wu

Time: August 13, 2:10-4:00pm

Name: ________________________

SID: ________________________

Problems	1	2	3	4	5	6	7	8	Total
Grades									
Out of	20	10	10	10	10	15	15	10	100

1. *This is a closed-book, closed-notes examination. So please put away your textbook, notebook, homework, quizzes, etc.*
2. *NO calculator is allowed.*
3. *Please be precise when writing your answers and use complete sentences when appropriate, especially in proofs.*

1. (5 points $\times$ 4) Multiple choices. For each of the following problems, there is only one correct answer. Please write your answers in the following table. Anything outside the table will not be graded.

Problem	(1)	(2)	(3)	(4)
Answer				

(1) Which of the following sets has a finite supremum and minimum, but no maximum:

(A) $[2, +\infty)$ (B) $\{r \in \mathbb{Q} : 0 \leq r < \sqrt{2}\}$

(C) $\mathbb{N}$ (D) $\{\frac{p}{q} : p = 1, 2$ or $3,$ and $q = 3, 4,$ or $5\}$

(2) Which of the following sequences converge?

(A) $\sin \sqrt{n}$ (B) $\frac{n!}{2^n}$

(C) $\left(\frac{1}{3+(-1)^n}\right)^n$ (D) $\left(1 + \frac{1}{n}\right)^{n^2}$

You may use the space below as scratch paper.

(3) Which of the following statements is false?

(A) If $\limsup s_n = \liminf s_n \in \mathbb{R}$, then the sequence (s_n) converges.

(B) $\limsup s_n = \sup S$, where S is the set of subsequential limits of the sequence (s_n).

(C) Every rational number is also algebraic.

(D) For any two sequences (s_n) and (t_n),

$$\limsup(s_n t_n) = \limsup s_n \cdot \limsup t_n.$$

(4) Which of the following statements is true?

(A) If $\sum a_n x^n$ has interval of convergence $[-R, R]$, then $\sum n a_n x^{n-1}$ also has interval of convergence $[-R, R]$.

(B) If $f : (a, b) \to \mathbb{R}$ is strictly increasing and differentiable on (a, b), then $f'(x) > 0$ for all $x \in (a, b)$.

(C) The function

$$H(x) = \begin{cases} 1, & -1 \le x \le 0 \\ 0, & 0 < x \le 1 \end{cases}$$

is integrable on $[-1, 1]$.

(D) If f is a real-valued function and $f^{(n)}(0)$ exist for all $n = 0, 1, 2, \cdots$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

You may use the space below as scratch paper.

2. (a) (5 points) Find the derivative $f'(x)$ of the function

$$f(x) = \cos(x^2 + e^{1/x})$$

(b) (5 points) Evaluate the following limit

$$\lim_{x \to 0^+} x^{\sqrt{x}}.$$

3. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 - x^2, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is continuous at $x = 1$.

4. (a) (4points) State any version of Fundamental Theorem of Calculus.

(b) (6points) Find

$$\lim_{x \to 0} \frac{1}{x} \int_0^x \sin(t^2)dt.$$

5. (10 points) Find the interval of convergence of the following series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{3^n n \ln n} x^n$$

6. (a) (5 points) State the Weierstrass M-test.

(b) (10 points) Suppose $a \in (0, 1)$. Show that the series

$$\sum_{n=0}^{\infty} \frac{x^n}{1 + \sin x}$$

converges uniformly on $[0, a]$.

7. (a) (5 points) State the Mean Value Theorem.

(b) (10 points) Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Show that if $f(a) = f(b) = 0$, then there exists $x_0 \in (a, b)$, such that

$$f(x_0) + f'(x_0) = 0.$$

Hint: Apply the mean value theorem to $h(x) = f(x)e^x$.

8. (10 points) Let $\sum a_n$ be a convergent series and (f_n) be a sequence of real-valued functions defined on $S \subset \mathbb{R}$ such that

$$|f_{n+1}(x) - f_n(x)| < a_n, \quad \forall n \in \mathbb{N}, \forall x \in S.$$

Prove that (f_n) is uniformly Cauchy on S and hence it is uniformly convergent on S.