EE 128 Fall 2003 Midterm Solutions

1) a) Let $x_1 = p$ $\quad \dot{x}_1 = x_2$

$x_2 = \dot{p}$ $\quad \dot{x}_2 = x_1 \cos(x_2) + x_3(1+u) \equiv f$

$x_3 = q$ $\quad \dot{x}_3 = x_1(x_1^2 - x_3) \equiv g$

$y = x_1(x_3 + u) \equiv h$

At equilibrium, ① $\dot{x}_1 = \boxed{x_2 = 0}$

② $\dot{x}_2 = 0 \rightarrow x_1 \cos(0) + x_3 = 0$

$x_1 + x_3 = 0$

③ $\dot{x}_3 = 0 \rightarrow x_1(x_1^2 - x_3) = 0$

From ②, $x_1 = -x_3$

From ③, $x_1(x_1^2 + x_1) = 0$

$x_1^2(x_1 + 1) = 0 \rightarrow$ either $\boxed{x_1 = -1}$

or $x_1 = 0$

↑ gives $(0,0,0)$ equilib

$\therefore \boxed{x_3 = 1}$

equilibrium: $\boxed{(x_1, x_2, x_3) = (-1, 0, 1)}$

b) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} \approx \begin{bmatrix} 0 & 1 & 0 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} \end{bmatrix}\Bigg|_{\text{equilib}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial f}{\partial u} \\ \frac{\partial g}{\partial u} \end{bmatrix} u$$

$$\frac{\partial f}{\partial x_1} = \cos(x_2)\Big|_{x_2=0} = 1$$

$$\frac{\partial f}{\partial u} = x_3\Big|_{x_3=1} = 1$$

$$\frac{\partial f}{\partial x_2} = -x_1 \sin x_2 \Big|_{\substack{x_1=-1 \\ x_2=0}} = 0$$

$$\frac{\partial f}{\partial x_3} = 1+u\Big|_{u=0} = 1$$

$$\frac{\partial g}{\partial x_1} = 3x_1^2 - x_3\Big|_{\substack{x_1=-1 \\ x_3=1}} = 3-1 = 2$$

$$\frac{\partial g}{\partial x_2} = 0 \qquad \frac{\partial g}{\partial u} = 0$$

$$\frac{\partial g}{\partial x_3} = -x_1\Big|_{x_1=-1} = 1$$

and $$y \approx \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \left(\frac{\partial h}{\partial u}\Big|_{\text{equi}}\right) u$$

$$\frac{\partial h}{\partial x_1} = x_3 + u\Big|_{\substack{x_3=1 \\ u=0}} = 1 \qquad \frac{\partial h}{\partial x_3} = x_1\Big|_{x_1=-1} = -1$$

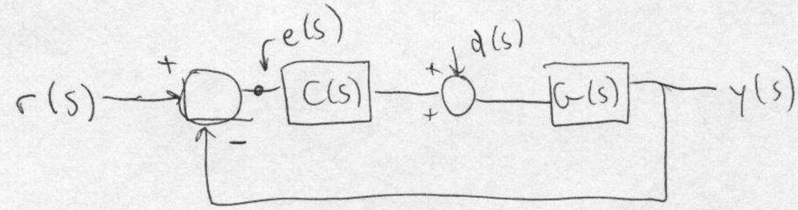
$$\frac{\partial h}{\partial x_2} = 0 \qquad \frac{\partial h}{\partial u} = x_1\Big|_{x_1=-1} = -1$$

$\therefore$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + (-1) u$$

2)



a) setting $d(s)=0$, transfer function from r to y is

$$\frac{Y(s)}{R(s)} = \frac{CG}{1+CG}$$

however, to be of type k, one must find the input to error transfer function (e(s) is error)

$$r - y = e$$

$$R(s) - R(s)\frac{CG}{1+CG} = E(s)$$

$$R(s)\left(1 - \frac{CG}{1+CG}\right) = E(s)$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+CG}$$

to be type k,

$$\lim_{s \to 0} s \frac{1}{1+C(s)G(s)} \cdot \frac{1}{s^{k+1}}$$ must be some constant (not 0, not ∞)

$$\lim_{s \to 0} \frac{1}{s^k + s^k C(s)G(s)} = X$$ (some constant)

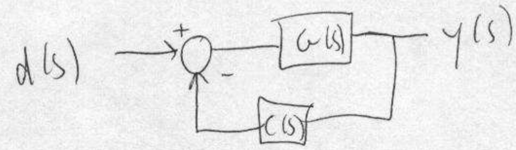
this is 0 in the limit

this must be some constant!

$\therefore$ $C(s)G(s)$ must have k more poles at the origin then it has zeros at the origin

b) This is False

Set $r=0$, with the reference at 0, the error is just th output y



$$\frac{Y(s)}{d(s)} = \frac{G(s)}{1+C(s)G(s)}$$

$$\lim_{s \to 0} s \frac{G(s)}{1+C(s)G(s)} \frac{1}{s^{k+1}}$$ must be a constant (not or ∞) for this to be t

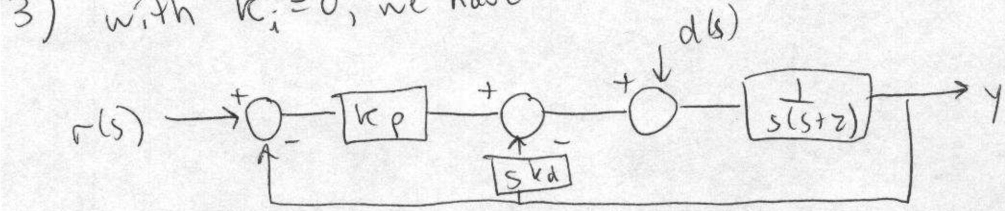
$$\text{so } \lim_{s\to 0} \frac{G(s)}{\underbrace{s^k}_{\text{goes to } 0} + s^k C(s)G(s)} = X \quad \text{(some constant)}$$

$$= \lim_{s\to 0} \frac{\cancel{G(s)}}{s^k C(s)\cancel{G(s)}} = \lim_{s\to 0} \frac{1}{s^k C(s)} = X$$

$\therefore$ $C(s)$ must have k more poles than it has zeros at the origin (both poles and zeros at the origin that is)

But before, our condition was on $G(s)\,C(s)$, so our poles at 0 could have come from $G(s)$, not $C(s)$, so this is <u>false</u>.

3) with $k_i = 0$, we have



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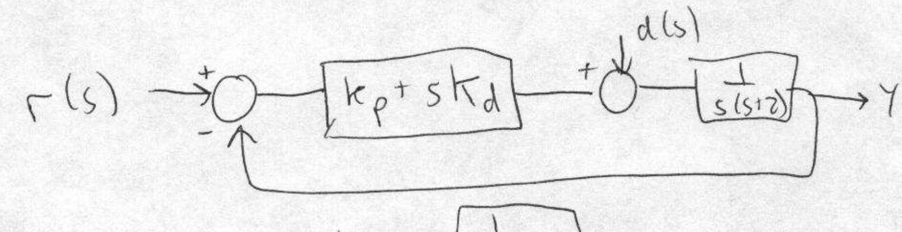
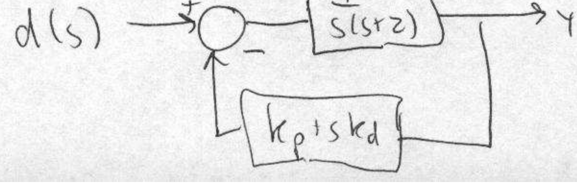


with $r(s) = 0$,


$$\frac{y(s)}{d(s)} = \frac{\frac{1}{s(s+2)}}{1 + \frac{k_p + sk_d}{s(s+2)}} = \frac{1}{s(s+2) + k_p + sk_d}$$

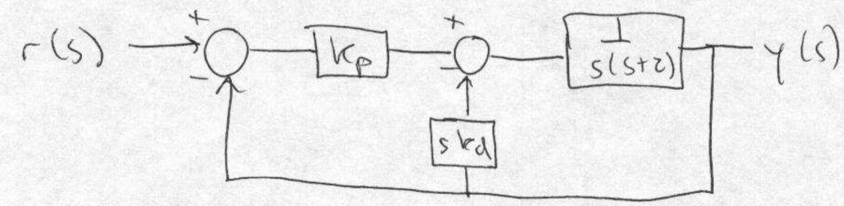
$$\frac{y(s)}{d(s)} = \frac{1}{s^2 + (2+k_d)s + k_p}$$

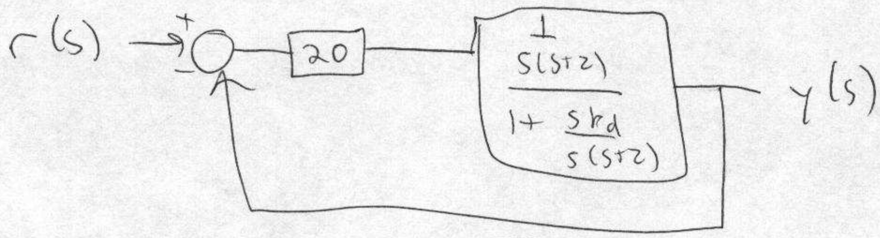
at steady state, unit step on $d(s)$,

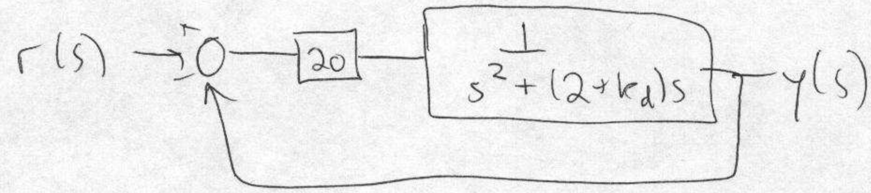
$$y(s) = 0.05 = \lim_{s \to 0} \cancel{s} \cdot \frac{1}{s^2+(2+k_d)s+k_p} \cdot \frac{1}{\cancel{s}}$$

$$0.05 = \frac{1}{k_p} \quad \therefore \quad \boxed{K_p = 20}$$

now with $d(s)=0$,







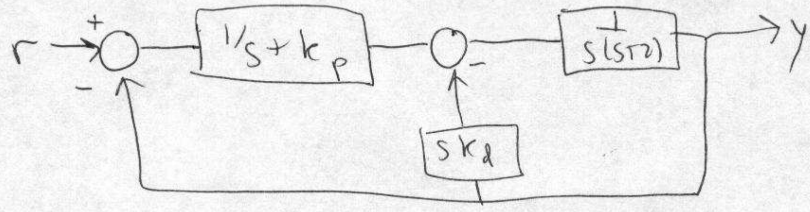
$$\frac{y(s)}{r(s)} = \frac{\frac{20}{s^2+(2+k_d)s}}{1+\frac{20}{s^2+(2+k_d)s}} = \boxed{\frac{20}{s^2+(2+k_d)s+20}}$$

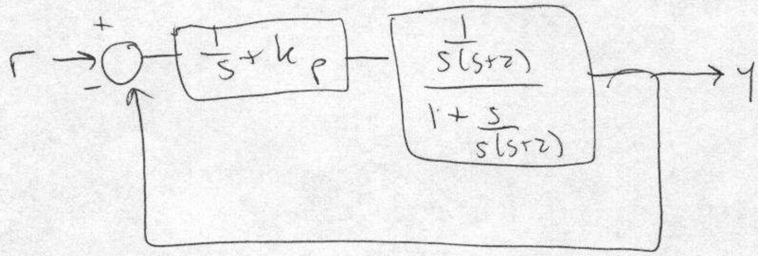
so $\quad 2+k_d = 2\xi w_n \qquad w_n^2 = 20$

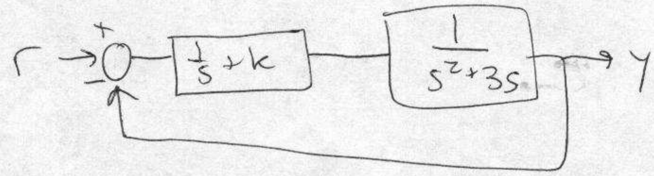
$$w_n = \sqrt{20}$$

$$k_d = 2(.7)(\sqrt{20})-2 = \boxed{4.26}$$

3 b)







$$\frac{y(s)}{r(s)} = \frac{\left(\frac{1}{s}+k\right)\left(\frac{1}{s^2+3s}\right)}{1+\left(\frac{1}{s}+k\right)\left(\frac{1}{s^2+3s}\right)}$$

$$= \frac{\frac{1}{s}+k}{s^2+3s+\frac{1}{s}+k} = \boxed{\frac{1+ks}{s^3+3s^2+ks+1}}$$

poles must be in left hand plane → use Routh's

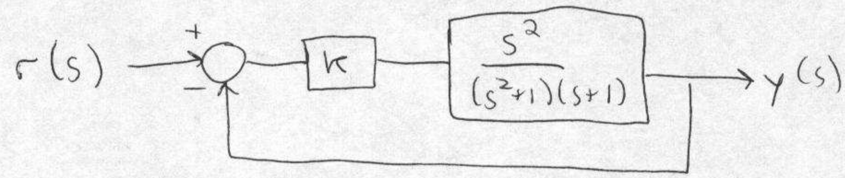
s^3	1	k
s^2	3	1
s^1	$\frac{3k-1}{3}$	0
s^0	1	

so $\frac{3k-1}{3} > 0$

$3k-1 > 0$

$\boxed{k > 1/3}$

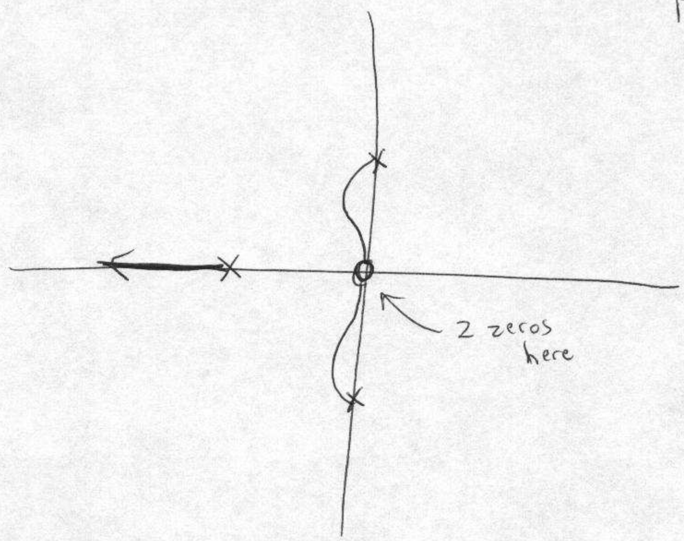
4)



root locus:

poles: $-1, \pm j$

zeros: 2 at 0



Departure angle for $+j$:

$$-45^\circ - 90^\circ - \theta + \overbrace{90^\circ + 90^\circ}^{\text{2 zeros at 0}} = 180^\circ$$

$$\boxed{\theta = -135^\circ}$$

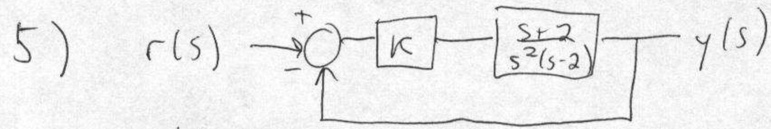
Arrival angle at zeros:

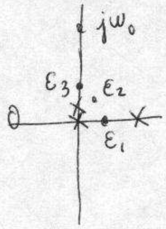
$$(-90^\circ) - (-90^\circ) + 2\theta = 180^\circ$$

$$\boxed{\theta = 90^\circ}$$

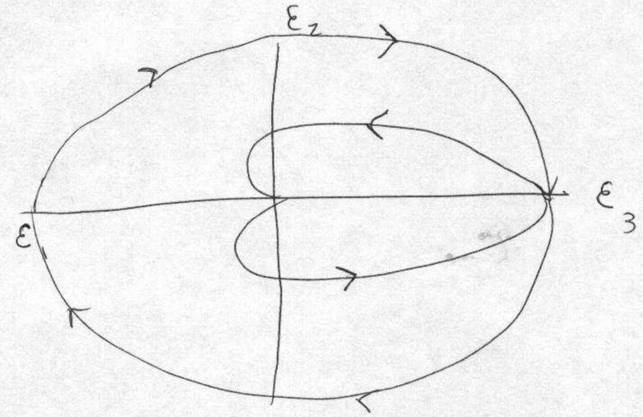
For stability, use a small k (actually any positive k will work)

5)





$j\omega$	$\lvert\cdot\rvert$	$\angle$
ε_1	∞	$-180°$
ε_2	∞	$-180°-45°-45° = -270°$
ε_3	∞	$-180°-90°-90° = 0°$
$j\omega_0$	C	$-135°-90°-90°+45° = -270°$
∞	0	$-180°$



$Z = N + P$ ← unstable pole = 1

↑ CW encircle = 1

$Z = 2$

∴ system unstable for all positive k

(note: can't use final value thm on this to check because it's unstable!)