Final Exam Solutions

Physics 7B
Lecture 1

Fall 2015

Problem 1

a) (5 points total)

$$engine\ Efficiency = \frac{W_{net}}{Q_{in}}\ (1\ points)$$

$Based\ on\ the\ first\ law\ \Delta E_{net} = 0\ for\ a\ cycle\ and\ therefore\ W_{net} = Q_{net}$ so:

$$engine\ Efficiency = \frac{Q_{net}}{Q_{in}} = \frac{Q_{in} - Q_{out}}{Q_{in}} = 1 - \frac{Q_{out}}{Q_{in}}$$

$$\eta_O = 1 - \frac{Q_{DA}}{Q_{BC}} = 1 - \frac{nC_V(T_D - T_A)}{nC_V(T_C - T_B)} = 1 - \frac{(T_D - T_A)}{(T_C - T_B)}\ (2\ points)$$

$$\eta_D = 1 - \frac{Q_{HE}}{Q_{FG}} = 1 - \frac{nC_V(T_E - T_H)}{nC_P(T_G - T_F)} = 1 - \frac{C_P(T_E - T_H)}{C_V(T_G - T_F)} = 1 - \frac{1(T_E - T_H)}{\gamma(T_G - T_F)}\ (2\ points)$$

b) (5 points total)
From adiabatic process CD:

$$P_C V_C{}^{\gamma} = P_D V_D{}^{\gamma}\ (1\ point)$$

$$nRT_C V_C{}^{\gamma-1} = nRT_D V_D{}^{\gamma-1}$$

$$T_C V_C{}^{\gamma-1} = T_D V_D{}^{\gamma-1}\ (1\ point)$$

$$T_C = T_D \frac{V_D{}^{\gamma-1}}{V_C{}^{\gamma-1}}$$

Since $V_C = V_B\ and\ V_D = V_A$

$$T_C = T_D \frac{V_A{}^{\gamma-1}}{V_C{}^{\gamma-1}}$$

also from adiabatic process AB:

$$P_B V_C{}^{\gamma} = P_A V_D{}^{\gamma}\ (1\ point)$$

$$T_B = T_A \frac{V_A{}^{\gamma-1}}{V_C{}^{\gamma-1}}\ (1\ point)$$

$$\eta_O = 1 - \frac{\left(T_C \frac{V_C{}^{\gamma-1}}{V_A{}^{\gamma-1}} - T_B \frac{V_C{}^{\gamma-1}}{V_A{}^{\gamma-1}}\right)}{(T_C - T_B)} = 1 - \frac{V_C{}^{\gamma-1}}{V_A{}^{\gamma-1}} = 1 - \left(\frac{V_C}{V_A}\right)^{\gamma-1}\ (1\ point)$$

c)

From adiabatic process GH: (1 *point*)

$$P_G V_G{}^{\gamma} = P_H V_H{}^{\gamma}$$
$$nRT_G V_G{}^{\gamma-1} = nRT_H V_H{}^{\gamma-1}$$

$$T_G V_G{}^{\gamma-1} = T_H V_H{}^{\gamma-1}$$
$$T_G = T_H \frac{V_H{}^{\gamma-1}}{V_G{}^{\gamma-1}}$$

where $V_H = V_E$ (1 *point*)

$$T_G = T_H \left(\frac{V_E}{V_G}\right)^{\gamma-1}$$

similarly

$$T_F = T_E \left(\frac{V_E}{V_F}\right)^{\gamma-1}$$

$$\eta_D = 1 - \frac{1(T_E - T_H)}{\gamma(T_G - T_F)} = 1 - \frac{T_G \left(\frac{V_G}{V_E}\right)^{\gamma-1} - T_F \left(\frac{V_F}{V_E}\right)^{\gamma-1}}{\gamma(T_G - T_F)}$$

now we need to relate T_G and T_F (1 *point*)

$$\frac{P_G V_G}{T_G} = \frac{P_F V_F}{T_F}$$

since $P_G = P_F$ (1 *point*)

$$T_F = \frac{P_F V_F}{P_G V_G} T_G = \frac{V_F}{V_G} T_G$$

$$\eta_D = 1 - \frac{T_G \left(\frac{V_G}{V_E}\right)^{\gamma-1} - \frac{V_F}{V_G} T_G \left(\frac{V_F}{V_E}\right)^{\gamma-1}}{\gamma\left(T_G - \frac{V_F}{V_G} T_G\right)} = \eta_D = 1 - \frac{V_G{}^{\gamma-1} V_E{}^{1-\gamma} - V_G{}^{-1} V_F{}^{\gamma} V_E{}^{1-\gamma}}{\gamma\left(1 - V_F{}^{1} V_G{}^{-1}\right)}$$

multiply top and bottom by $\frac{V_G}{V_E}$

$$\eta_D = 1 - \frac{{V_G}^{\gamma}\,{V_E}^{-\gamma} - {V_F}^{\gamma}\,{V_E}^{-\gamma}}{\gamma\left(\frac{V_G}{V_E} - \frac{V_F}{V_E}\right)} = 1 - \frac{\left(\frac{V_G}{V_E}\right)^{-\gamma} - \left(\frac{V_E}{V_F}\right)^{-\gamma}}{\gamma\left(\left(\frac{V_E}{V_G}\right)^{-1} - \left(\frac{V_E}{V_G}\right)^{-1}\right)}$$

$(1\ point)$

Prof. Birgeneau Final Problem 2 grading rubrics

(a) Magnitude

Each of the charge will generate an electric field with magnitude

$$E = \frac{q}{4\pi\varepsilon_0}\frac{1}{(x^2 + (\frac{d}{2})^2)}$$ ....2 pts

Then do the vector sum

X component cancels out, only y component remains, so the direction of the electric field generated by the dipole is in **y direction** 1.5pts

The magnitude of the total electric field is

$$E = \frac{q}{4\pi\varepsilon_0}\frac{1}{(x^2 + (\frac{d}{2})^2)}\left(2\times\frac{d/2}{\sqrt{x^2 + (\frac{d}{2})^2}}\right) = \frac{qd}{4\pi\varepsilon_0 (x^2 + (\frac{d}{2})^2)^{3/2}}$$ 2pts

In the limit x>>d,

$$E = \frac{qd}{4\pi\varepsilon_0 x^3} = \frac{p}{4\pi\varepsilon_0 x^3}$$ 2pts

(P = qd is worth 1pt, simplification is worth 1 pt)

(b) $$\vec{B} = \frac{\mu_0 I}{4\pi}\int\frac{d\vec{l}\times\hat{r}}{r^2}$$ （No pts for this because this is in the formula sheet）

By symmetry, the magnetic field will **point in the x direction**, so we pick that component in our application of the Biot-Savart Law..1.5pts

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int\frac{dl}{x^2+R^2}\frac{R}{(x^2+R^2)^{1/2}}\hat{x} \quad \frac{\mu_0 I}{4\pi}\frac{R}{(x^2+R^2)^{3/2}}\int dl\hat{x}$$ 2pts

$$\vec{B} = \frac{\mu_0 I}{2\pi}\frac{\pi R^2}{(x^2+R^2)^{3/2}}\hat{x}$$ 2pts

In the limit x>>R,

$$\vec{B} = \frac{\mu_0}{2\pi}\frac{\mu}{(x^2+R^2)^{3/2}}\hat{x} = \frac{\mu_0}{2\pi}\frac{\mu}{x^3}\hat{x}$$ 2pts

(µ=I*Pi*R^2 is worth 1 pt, the simplification is worth another 1pt)

(3)

(a) The electric field is

$$\begin{aligned}\vec{E} &= -\nabla V && \text{(2.5 Marks)}\\ &= -\frac{\partial V}{\partial r}\hat{r} \\ &= \frac{q}{4\pi\epsilon_0 r}e^{-r/a}\left(\frac{1}{a}+\frac{1}{r}\right)\hat{r} && \text{(2.5 Marks)}\end{aligned}$$

(b) The flux is

$$\begin{aligned}\Phi_E &= \oint \vec{E}.d\vec{A} && \text{(1 Mark)}\\ &= |\vec{E}|\oint |d\vec{A}| \\ &= |\vec{E}|(4\pi r^2) && \text{(2 Marks)}\\ &= \frac{q}{\epsilon_0}e^{-r/a}\left(1+\frac{r}{a}\right) && \text{(2 Marks)}\end{aligned}$$

(c) Gauss' Law tells us that

$$\Phi_E = \oint \vec{E}.d\vec{A} = Q_{enc}/\epsilon_0 \qquad \text{(2 Marks)}$$

So the enclosed charge is

$$\begin{aligned}Q_i &= \epsilon_0 \Phi_E \\ &= qe^{-r/a}\left(1+\frac{r}{a}\right) && \text{(1 Mark)}\end{aligned}$$

In the limit $r \to \infty$ we have $Q_i \to 0$. This implies that the total charge of the hydrogen atom is 0 (neutral), which makes sense because it consists of a proton and an electron. (2 Marks)

Problem 4

a) For both the interior and exterior of the wire, we choose our Amperian Loop C to be a circle of radius r centered on the wire's center. From the symmetry of the configuration, the magnetic field has constant magnitude along C, so Ampere's Law

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$$

implies that

$$2\pi r B = \mu_0 I_{enc} \qquad \textbf{1 Point}$$

where $B = |\vec{B}|$. Our task is now reduced to finding I_{enc} for $r < \rho$ and $r > \rho$.

For $r < \rho$,

$$I_{enc} = \int_0^r j(r') \cdot 2\pi r' dr' = \int_0^r 2\pi m r'^3 dr' = \frac{2\pi m r^4}{4} \qquad \textbf{1 Point}$$

thus

$$B = \frac{\mu_0 m r^3}{4} \qquad \textbf{1 Point}$$

For $\rho < r$,

$$I_{enc} = \int_0^\rho j(r') \cdot 2\pi r' dr' = \frac{2\pi m \rho^4}{4} \qquad \textbf{1 Point}$$

so

$$B = \frac{\mu_0 m \rho^4}{4r} \qquad \textbf{1 Point}$$

If you orient the wire so that the current is coming directly at you, then $\vec{B} = B(r)\hat{\phi}$, which can be shown using the right-hand rule.

b) As mentioned above, if the current is coming directly at you, then the magnetic field circulates in the counter clockwise direction. The magnitude increases as r increases in the region $0 < r < \rho$ but the magitude decreases with increasing r in the region $\rho < r < \infty$. **1 Point** for correct magnitude behavior. **1 Point** for correct direction.

c) To find induced current, we must use Faraday's Law

$$\mathcal{E} = \oint_C \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

If the distance from the center of the wire and the left edge of the loop is x, then the flux is given by

$$\Phi_B = b\int_x^{x+a} B(r)dr = \frac{\mu_0 m b \rho^4}{4}\int_x^{x+a} \frac{1}{r}dr = \frac{\mu_0 m b \rho^4}{4} \ln\left(1 + \frac{a}{x}\right) \quad \textbf{3 Points}$$

i) If we move the loop in the y direction, the flux does not change with time. Therefore, if the induced current is I, we have $I = \mathcal{E}/R = 0$ **2 Points**

ii) Instead, if we move the loop in the x direction, x in the flux equation above is a function of time such that $\frac{dx}{dt} = v$, so x = x(t) = b+vt. Calculating the induced current through Faraday's

Law yields

$$I = \frac{\mathcal{E}}{R} = \frac{\mu_0 m b \rho^4}{4R} \cdot \frac{x}{x+a} \cdot \frac{a}{x^2} \cdot v = \frac{\mu_0 m b v a \rho^4}{4Rx(x+a)} \qquad \textbf{3 Points}$$

Solution

(a) The flux through the solenoid (when a current I is flowing) is

$$\begin{aligned}\Phi_B &= BA \\ &= N \times \left(\frac{\mu_0 NI}{l}\right) A \\ &= \frac{\mu_0 N^2 A}{l} I\end{aligned}$$

Right away we can see that the self-inductance is

$$L = \boxed{\frac{\mu_0 N^2 A}{l}.}$$

(b) We know that the energy is given by

$$\begin{aligned}U &= \int P dt \\ &= \int I\mathcal{E} dt \\ &= \int_0^t I(t') \left(L\frac{dI(t')}{dt'}\right) dt' \\ &= L\int_0^I I' dI' \\ &= \boxed{\frac{1}{2}LI^2.}\end{aligned}$$

(c) The volume of the solenoid is $V = Al$, so the energy density is

$$\begin{aligned}u &= \frac{U}{V} \\ &= \frac{\frac{1}{2}LI^2}{Al} \\ &= \frac{\mu_0 N^2 I^2}{2l^2}.\end{aligned}$$

We know that the magnetic field is $B = \mu_0 nI$, so we can write this as

$$u = \boxed{\frac{1}{2\mu_0}B^2.}$$

Rubric

(a)
- **1 Point** for the correct magnetic field.
- **1 Point** for reasonable attempt at magnetic flux.
- **1 Point** for correct magnetic flux, including N loops.
- **1 Points** for self-inductance.

(b)
- **1 Point** for correct emf of inductor.
- **1 Point** for correct integral setup.
- **1 Point** for correct answer.

(c)
- **1 Point** for correct setup of energy density.
- **2 Points** for correct answer, including work shown.

Problem 6

a) Applying Kirchhoff's loop rule we get:

$$V_0 - L\frac{dI}{dt} - RI = 0 \Leftrightarrow \frac{dI}{dt} = -\frac{R}{L}I + \frac{V_0}{L} \quad (1)$$

Where we have manipulated the equation to bring it to the form it appears in the equation sheet. The solution given is:

$$I(t) = \frac{V_0}{R}(1 - e^{-At}) + I(0)e^{-At} = \frac{V_0}{R}(1 - e^{-\frac{t}{\tau}}), \quad (2)$$

where we have used that the initial current $I(0) = 0$ and that $A = \frac{R}{L} = \tau^{-1}$ is the inverse of the time constant.

b) This part asks us to justify our assumption that $I(0) = 0$. When the switch is open there can be no current, and because there is an inductor in the system, the current cannot change discontinuously. Therefore, the current is still zero immediately after closing the switch.

c) As found in part a), $\tau = \frac{L}{R} = 0.01s$.
d) The current is monotonically increasing in time, starting at zero and asymptoting its maximum value at infinity. Plugging in $t = \infty$ in equation (2), we see that $I_{max} = \frac{V_0}{R} = 1A$.
This same answer can also be reached by arguing that at late times, when the system reaches a steady state and the current becomes constant, the inductor acts as a wire, so by Ohm's Law $V_0 = I_{max}R$, which implies $I_{max} = \frac{V_0}{R}$.

e) The power delivered by the battery is always given by $P = IV_0$. Therefore, at the maximum current we have $P_{max} = I_{max}V_0 = \frac{V_0^2}{R} = 10W$.

f) The energy stored in an inductor is given by $U_L = \frac{1}{2}LI^2$, which means that the power (rate of energy being stored) is given by $P_L = \frac{dU_L}{dt} = LI\frac{dI}{dt} = 0$, since the current becomes constant at late times.

(7.a) The principle needed to solve this problem is conservation of energy applied in the form of Kirchhoff's loop rule. By summing the voltages across each circuit element, we may write down the differential equation directly. $\frac{q}{C} - L\frac{dI}{dt} = 0$.

(7.b) The critical point here is to realize that the rate at which charge is drained from the capacitor is related to the current. Using a negative sign to correct for the fact that $\frac{dq}{dt}$ is negative, we have $I = -\frac{dq}{dt}$. This implies that $\frac{dI}{dt} = -\frac{d^2q}{dt^2}$, which may be substituted into the previous differential equation, with the following result.

$$\frac{q}{C} + L\frac{d^2q}{dt^2} = 0 \Rightarrow \boxed{\frac{d^2q}{dt^2} = \frac{-q}{LC}}$$

The boxed equation is the correct form, as it follows the harmonic oscillator form. This equation suggests that the charge on the capacitor oscillates in time, but there are a number of other quantities which oscillate in the circuit (e.g. voltage across the capacitor, current in the circuit, energy stored in the inductor). Students may have knowledge of this behavior from their studies, so any quantity which does oscillate in the circuit will be counted as correct.

(7.c) Recognizing that the harmonic oscillator equation has the form $\frac{d^2x}{dt^2} = -\omega^2 x$, we may infer that $\omega = \sqrt{\frac{1}{LC}}$. However, the formula for the frequency of an LC circuit is a formula which is well-known, and students may be able to simply cite the result. Furthermore, given that the problem asserts that the oscillation frequency may be written in terms of L and C, it is acceptable for students to derive the correct answer through dimensional analysis alone.

(7.d) Using the formula sheet, we find that the differential equation describing the charge on the capacitor has the solution

$$q(t) = q_{max}Cos(\frac{t}{\sqrt{LC}} + \delta)$$

The previously determined relationship $I = -\frac{dq}{dt}$ means that $I(t)$ is given by

$$I(t) = -\frac{d}{dt}q_{max}Cos(\frac{t}{\sqrt{LC}} + \delta) = \frac{q_{max}}{\sqrt{LC}}Sin(\frac{t}{\sqrt{LC}} + \delta)$$

It remains to determine the values of q_{max} and δ using the initial conditions.

$$q(t=0) = Q_0$$
$$q_{max}Cos(\delta) = Q_0$$

$$I(t=0) = 0$$
$$\frac{q_{max}}{\sqrt{LC}}Sin(\delta) = 0 \Rightarrow \delta = 0, q_{max} = Q_0$$

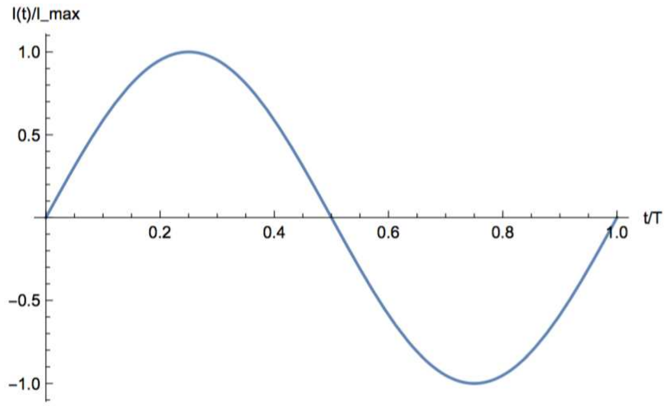
The correct formula for the current as a function of time is therefore

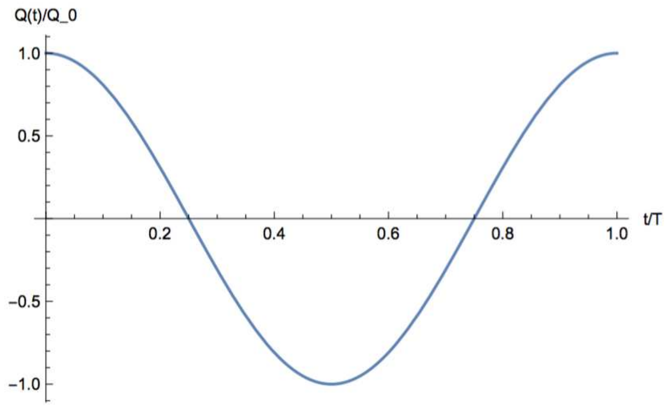
$$I(t) = \frac{Q_0}{\sqrt{LC}}Sin(\frac{t}{\sqrt{LC}})$$

(7.e) The previous part allows students to write the charge as a function of time,

$$q(t) = Q_0 Cos(\frac{t}{\sqrt{LC}}) = Q_0 Sin(\frac{t}{\sqrt{LC}} + \frac{\pi}{2})$$

Both functions are then sinusoidal and out of phase by $\frac{\pi}{2}$.





(7.f) The energies stored in a capacitor and an inductor are $\frac{q^2}{2C}$ and $\frac{1}{2}LI^2$, respectively. Using the previously mentioned formulas for the charge and currents as functions of time, we obtain the following results.

$$\begin{aligned} U_E(t) &= \frac{1}{2C}(q(t))^2 \\ &= \frac{Q_0^2}{2C} Cos^2(\frac{t}{\sqrt{LC}}) \end{aligned}$$

$$\begin{aligned} U_B(t) &= \frac{1}{2}L(I(t))^2 \\ &= \frac{1}{2}L\frac{Q_0^2}{LC} Sin^2(\frac{t}{\sqrt{LC}}) \\ &= \frac{Q_0^2}{2C} Sin^2(\frac{t}{\sqrt{LC}}) \end{aligned}$$

(7.g) Summing the formulas in the previous part, we have

$$\begin{aligned} U_{total}(t) &= U_E(t) + U_B(t) \\ &= \frac{Q_0^2}{2C}Cos^2(\frac{t}{\sqrt{LC}}) + \frac{Q_0^2}{2C}Sin^2(\frac{t}{\sqrt{LC}}) \\ &= \frac{Q_0}{2C} \end{aligned}$$

The total energy in the system is constant in time, which reflects the fact that energy is conserved (no dissipative effects such as resistance are present in the system). In fact, students who recognize that energy is conserved may simply state that the total energy is just the initial energy stored in the capacitor and is constant in time.

Rubric

In dealing with LC circuits, students may be expected to have some qualitative knowledge in the behavior of charge, current, and energy within the system. For this problem, the solution of each piece provides a clue to solving subsequent parts. However, in an attempt to treat each part independently so that students can get credit for correct answers when they have not solved all parts, I have tried to recognize instances in which students may use their experience with the circuits to generate solutions. For instance, students may not correctly solve the second part which requires them to write a harmonic oscillator equation for the charge in the system. They may know, though, that the charge and current in an LC circuit oscillate as sinusoidal functions, which they may use to solve the fourth part instructing readers to give the current as a function of time. Such a solution will be counted as valid although no logical precedence for a sinusoidal function has been introduced.

In some of these instances, I require no explicit statement of knowledge from the student and consider the answer to represent his or her acquired understanding of the subject. In others, I expect some brief explanation of the answer when the derivation has been omitted; I have tried to note these cases and the explanations I will accept below.

Sign errors are given a half point deduction when they occur, except in cases in which they have a negligible effect on the answer given. Propagation of mistakes made in one part to work in following parts is penalized only when the results are so opposed to the behavior of the circuit that students should have recognized the error. For example, finding the maximum current in the fourth part may lead to an incorrect statement of the energy in the inductor in the sixth part, but this is permissible because the function will still have the correct form. However, using this result in the final part may yield a time dependent expression for the total energy, which the student should know is incorrect based on the conservative nature of the circuit. Hence, the error would be penalized in the fourth and final parts but not the sixth.

(7.a) **(i)** 1 pt Statement that conservation of energy or Kirchhoff's loop rule is required to solve the problem

(ii) 1 pt Correct voltage drop across capacitor

(iii) 1 pt Correct voltage drop across inductor

(7.b) **(i)** .5 pt Recognition that $-\frac{dq}{dt} = I$

(ii) 1 pt Rearrangement of differential equation into harmonic oscillator form

(iii) .5 pt Correct statement of a quantity oscillating in the circuit. Although the question leads students to see that charge is oscillating, many other answers are acceptable. A (not exhaustive) list of other answers: current, voltage across the capacitor, energy in the inductor.

(7.c) **(i)** 2 pt Statement of the frequency. This may be taken by analogy from the harmonic oscillator equation, dimensional analysis using L and C, or simply by stating the well-known formula.

(7.d) **(i)** .5 pt Recognition that the solution is sinusoidal

(ii) .375 pt Correct statement of zero initial current

(iii) .375 pt Correct statement of initial charge

(iv) .375 pt Application of conditions to produce the correct amplitude

(v) .375 pt Application of conditions to produce the correct phase

(7.e) **(i)** .65 pt Correct depiction of initial charge condition (positive, maximum)

(ii) .1 pt Graph of charge oscillates in time

(iii) .65 pt Correct depiction of initial current condition (zero, inflection pt)

(iv) .1 pt Graph of current oscillates in time

(v) .5 pt Correct statement of phase relationship

(7.f) **(i)** .25 pt Correct statement of energy in a capacitor

(ii) .25 pt Correct statement of energy in an inductor

(iii) .75 pt Correct substitution of charge function into capacitor energy

(iv) .75 pt Correct substitution of current function into inductor energy

(7.g) **(i)** 1 pt Statement that energy is time independent due to energy conservation

(ii) 1 pt Correct addition of the energies in the capacitor and inductor