Physics 137A, Spring 2004 , Section 1 (Hardtke), Midterm II

You are allowed one 8 1/2 by 11 page (both sides) with notes. Put your answers on separate sheets or an exam book. Write your name on each page. There are four questions.

1. Suppose a 2×2 matrix U is unitary, i.e. $U^\dagger U = \mathbf{1}$.
   (a) Find the conditions this places on its components. [10 pts.]
   (b) It turns out that the S-matrix is unitary. Which of the above conditions correspond to $R+T=1$? R is the reflection coefficient and T is the transmission coefficient. [10 pts.]
2. Assume $\hat{A}$ is a linear operator with eigenvector $|\phi\rangle$ and eigenvalue a. Another linear operator $\hat{B}$ has the following commutation relation with $\hat{A}$:
   $$[\hat{A}, \hat{B}] = \hat{B} + 2\hat{B}\hat{A}^2$$
   Show that $\hat{B}|\phi\rangle$ is an eigenvector of $\hat{A}$ and find the eigenvalue. [15 pts.]
3. Given a wave function Ψ that satisfies the Schrödinger equation:
   $$\imath\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi,$$
   show that for any Hermitian operator $\hat{A}$,
   $$\imath\hbar\frac{d\langle\hat{A}\rangle}{dt} = \langle[\hat{A}, \hat{H}]\rangle + \imath\hbar\langle\frac{\partial\hat{A}}{\partial t}\rangle$$
   [25 pts.]
4. A 1-dimensional harmonic oscillator is initially in the state,
   $$|\Psi(x)\rangle = \frac{1}{\sqrt{5}}[2|\psi_2(x)\rangle + |\psi_3(x)\rangle],$$
   where $|\psi_n\rangle$ $(n = 0, 1, 2, \ldots)$ are the stationary state solutions to the potential $V(x) = \frac{1}{2}m\omega^2x^2$.
   (a) If you measure the energy of the system, what are the allowed energies and what is the probability of getting each energy? [10 pts.]
   (b) The raising operator $\hat{a}_+$, applied to a stationary-state $|\psi_n(x)\rangle$, gives:
   $$\hat{a}_+|\psi_n\rangle = c_n|\psi_{n+1}\rangle.$$
   Show that $c_n = \sqrt{(n+1)\hbar\omega}$. [25 pts.]
   Hints: Remember that $(\hat{a}_-)^\dagger = \hat{a}_+$ and consider the inner product:
   $$\langle\hat{a}_+\psi_n|\hat{a}_+\psi_n\rangle$$
   (c) If you measure the energy of the state given by $|\Phi\rangle = \hat{a}_+|\Psi(x)\rangle$ (with the expression for $|\Psi(x)\rangle$ given above), what are the allowed energies and what is the probability of each? [15 pts.]