Math 55 Second Midterm Exam, Prof. Srivastava
April 5, 2016, 3:40pm–5:00pm, F295 Haas Auditorium.

Name: ____________________

SID: ____________________

Instructions: Write all answers in the provided space. Please write carefully and clearly, in complete English sentences. This exam includes three pages of scratch paper at the end, which must be submitted, but will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page.

You are allowed to bring one letter-size single sided page of notes.

UC Berkeley Honor Code: *As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others.*

Sign here: ____________________

Question	Points
1	8
2	10
3	6
4	6
5	6
6	7
7	7
Total:	50

Do not turn over this page until your instructor tells you to do so.

1. Circle true (**T**) or false (**F**) for each of the following. There is no need to provide an explanation.

   (a) (2 points) If E and F are events in a probability space such that $p(F) \neq 0$ then

   $$p(E|F) \leq p(E).$$

   T **F**

   > **Solution:** False. Consider an event F with $p(F) = 1/2$ and $E = F$; then $p(E|F) = 1$ but $p(E) = 1/2$.

   (b) (2 points) If E and F are independent events in a probability space, then E and $\overline{F}$ are also independent.

   T **F**

   > **Solution:** True. We calculate
   >
   > $$p(E \cap \overline{F}) = p(E) - p(E \cap F) = p(E) - p(E)p(F) = p(E)(1 - p(F)) = p(E)p(\overline{F}),$$
   >
   > so E and $\overline{F}$ are independent.

   (c) (2 points) If $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$ are random variables such that $X(s) > Y(s)$ for all $s \in S$, then

   $$\mathbb{E}X > \mathbb{E}Y.$$

   T **F**

   > **Solution:** True. By the definition of expectation, we have
   >
   > $$\mathbb{E}X = \sum_{s \in S} p(s)X(s).$$
   >
   > Note that $p(s)X(s) \geq p(s)Y(s)$ for all s, since $p(s) \geq 0$. Moreover since $\sum_{s \in S} p(s) = 1$ at least one of these terms must have $p(s) > 0$ and thereby be a strict inequality. Summing over all s gives
   >
   > $$\sum_{s \in S} p(s)X(s) > \sum_{s \in S} p(s)Y(s),$$
   >
   > and the right hand side is just $\mathbb{E}Y$.

   (d) (2 points) If X and Y are independent random variables then

   $$\mathbb{E}(X + Y)^2 = (\mathbb{E}X + \mathbb{E}Y)^2.$$

T F

Solution: False. When $X = 0$ this becomes $\mathbb{E}Y^2 = (\mathbb{E}Y)^2$, which is not true for

$$Y = \begin{cases} +1 & \text{with prob.}1/2 \\ -1 & \text{with prob.}1/2 \end{cases}.$$

2. Nikhil is trying to solve a certain mathematical problem. There is a 2/3 chance that he will pass out while writing. If he passes out, there is a 3/4 chance that he will make an arithmetic mistake. If he doesn't pass out, there is only a 3/16 chance.

   (a) (3 points) What is the probability that Nikhil will make a mistake?

   Solution: Let F be the event that Nikhil passes out and let M be the event that he makes a mistake. We are given that $p(F) = 2/3$ as well as the conditional probabilities

   $$p(M|F) = 3/4 \qquad p(M|\overline{F}) = 3/16.$$

   By the law of total probability we have:

   $$p(M) = p(M|F)p(F) + p(M|\overline{F})p(\overline{F}) = \frac{3}{4} \cdot \frac{2}{3} + \frac{3}{16}\left(1 - \frac{2}{3}\right) = \frac{1}{2} + \frac{1}{16} = \frac{9}{16}.$$

   (b) (3 points) Given that he does make a mistake, what is the probability that he passed out?

   Solution: We use Bayes' rule:

   $$p(F|M) = \frac{p(M|F)p(F)}{p(M)} = \frac{(3/4)(2/3)}{(9/16)} = \frac{8}{9}.$$

(c) (4 points) Suppose now that in an alternate universe, Nikhil passes out with probability $q \in (0, 1)$ instead of $2/3$, with everything else unchanged — i.e., we still have probability $3/4$ of a mistake if he does pass out, and $3/16$ of a mistake if he doesn't pass out.

Is there some value of q for which the events $E = \{\text{Nikhil makes a mistake}\}$ and $F = \{\text{Nikhil passes out}\}$ are independent? If so, find such a q, and if not prove that no such q exists.

Solution: There is no such value of q. Here is a proof: Assume $0 < q < 1$ and E and F are independent. Observe that since $p(F) = q \neq 0$ and $p(\overline{F}) = 1 - q \neq 0$, both of the conditional probabilities $p(E|F)$ and $p(E|\overline{F})$ are well-defined, and must be equal to $3/4$ and $3/16$ by assumption. By question 1 part 2, E and $\overline{F}$ are also independent. Thus, we must have

$$p(E) = p(E|F) = 3/4$$

as well as

$$p(E) = p(E|\overline{F}) = 3/16,$$

which is absurd.

There is also a more algebraic proof: assume q is such that E and F are independent. By the law of total probability:

$$p(E) = p(E|F)q + p(E|\overline{F})(1-q) = \frac{12q + 3(1-q)}{16}.$$

By independence, we must have $p(E) = p(E|F) = 3/4$, so

$$\frac{12q + 3(1-q)}{16} = \frac{12}{16}.$$

Rearranging implies that $q = 1$, which is impossible since we assumed $q \in (0, 1)$.

3. (6 points) There are 5 distinguishable bins labeled $\{1, 2, 3, 4, 5\}$. How many ways are there of placing 100 indistinguishable balls into the bins, if each bin must have at least as many balls as its label? Be sure to explain your reasoning.

Solution: We can construct an allocation which meets the requirements as follows: (1) first place 1 ball in bin 1, 2 in bin 2,…, 5 in bin 5. There is only one way to do this since the balls are indistinguishable, and the total number of balls used is $1 + 2 + 3 + 4 + 5 = 15$.

(2) Place the remaining 85 balls into the 5 (distinguishable) bins with no constraints. This is the same as counting combinations with repetitions ("stars and bars"), for which we have the formula:

$$\binom{85 + 5 - 1}{5 - 1} = \binom{89}{4}.$$

Thus, by the product rule the total number of ways is $\binom{89}{4}$.

Name and SID: ______________________________

4. (6 points) Show using a combinatorial proof that:

$$\binom{2n}{3} = \binom{n}{3} + \binom{n}{3} + \binom{n}{2}n + n\binom{n}{2}.$$

Solution: Let us count the number of 3−element subsets of $A = \{1, \ldots, 2n\}$ in two different ways. On one hand, we know that this is just $\binom{2n}{3}$.

On the other hand, we can break the set into two disjoint sets of n elements each:

$$B = \{1, \ldots, n\} \quad \text{and} \quad C = \{n+1, \ldots, 2n\}.$$

To choose a 3−element subset $S \subseteq A$, we have the following mutually exclusive possibilities:

- choose three elements from B ($\binom{n}{3}$ ways)
- choose two elements from B and one from C ($\binom{n}{2}\binom{n}{1}$ ways, by the product rule)
- choose one element from B and two from C ($\binom{n}{1}\binom{n}{2}$ ways)
- choose all three elements from C ($\binom{n}{3}$ ways).

By the sum rule, the total number of ways is

$$\binom{n}{3} + \binom{n}{2}n + n\binom{n}{2} + \binom{n}{3}.$$

Since we have counted the same collection of objects in two different ways, we must have

$$\binom{2n}{3} = \binom{n}{3} + \binom{n}{3} + \binom{n}{2}n + n\binom{n}{2},$$

as desired.

5. (6 points) Show that any set of 76 distinct positive integers chosen from $\{1, \ldots, 100\}$ must contain 4 contiguous integers (i.e., $n, n+1, n+2, n+3$ for some n).

Solution: This is a pigeonhole problem. Consider the disjoint sets:

$$\{1,2,3,4\}, \{5,6,7,8\}, \ldots, \{97,98,99,100\}.$$

Each set contains four elements, their union is $\{1, \ldots, 100\}$, and there are exactly 25 of them. Let these sets be the boxes. Choosing 76 integers in $\{1, \ldots, 100\}$ is the same as putting 76 pigeons in these boxes; by the generalized pigeonhole principle, some box must contain $\lceil 76/25 \rceil = 4$ pigeons. But this is a continguous sequence of 4 integers, completing the proof.

You could also do this directly by contradiction. If there is no contiguous sequence, each of the subsets defined above must contain at most 3 *of the chosen elements, so the total is at most* $3 \times 25 = 75$, *a contradiction.*

Name and SID: ____________________

6. (7 points) Prove that for every positive integer $n \geq 1$, the sum

$$S_n = 1 + 3 + \ldots + (2n - 1)$$

is a perfect square.

Solution: The key here is getting the right induction hypothesis. Playing with a few examples, we see that:

$$S_1 = 1 = 1^2 \quad S_2 = 1 + 3 = 2^2 \quad S_3 = 1 + 3 + 5 = 3^3 \ldots,$$

so it is reasonable to conjecture the stronger statement:

$$S_n = 1 + 3 + \ldots + (2n - 1) = n^2$$

for all $n \geq 1$. Let $P(n)$ be the predicate that makes this assertion for a particular n. We proceed by induction.

Basis Step: We verify that $1 = 1^1$, so $P(1)$ is true.

Induction Step: Assume $k \geq 1$ and assume that $P(k)$ is true. We will show that $P(k + 1)$ is true. Observe that

$$\begin{aligned} S_{k+1} &= 1 + 3 + \ldots + (2k - 1) + 2(k + 1) - 1 \\ &= S_k + 2k + 2 - 1 \\ &= k^2 + 2k + 1 \quad \text{by the induction hypothesis} \\ &= (k + 1)^2, \end{aligned}$$

as desired, completing the inductive step.

7. (7 points) Consider the function recursively defined by:

$$f(1) = 3 \quad f(2) = 2 \quad f(3) = 1 \qquad f(k+1) = f(k) + f(k-1)f(k-2) \quad k \geq 3.$$

Prove that $f(n) \leq 2^{2^n}$ for all $n \geq 1$.

Solution: We proceed by strong induction. Let $P(n)$ be the predicate "$f(n) \leq 2^{2^n}$".

Basis Step: We verify that

$$f(1) = 3 \leq 2^2 \quad f(2) = 2 \leq 2^4 \quad f(3) = 1 \leq 2^8,$$

so $P(1), P(2)$, and $P(3)$ are true.

Induction Step: Assume $k \geq 3$ and assume that $P(1) \dots P(k)$ are true. Observe that by the recursive definition:

$$\begin{aligned}
f(k+1) &= f(k) + f(k-1)f(k-2) \\
&\leq 2^{2^k} + 2^{2^{k-1}} 2^{2^{k-2}} \quad \text{by the inductive hyp.} \\
\\
&= 2^{2^k} + 2^{2^{k-1}+2^{k-2}} \\
&\leq 2^{2^k} + 2^{2^k} \quad \text{since } 2^{k-1} + 2^{k-2} \leq 2 \cdot 2^{k-1} \leq 2^k \\
&= 2 \cdot 2^{2^k} \\
&= 2^{2^k+1}
\end{aligned}$$

We now observe that $2^k + 1 \leq 2^{k+1}$ whenever $k \geq 0$, so we conclude that

$$f(k+1) \leq 2^{2^{k+1}},$$

completing the inductive step.