Math 55 Second Midterm Exam, Prof. Srivastava
April 5, 2016, 3:40pm–5:00pm, F295 Haas Auditorium.

Name: _____

SID: _____

Instructions: Write all answers in the provided space. Please write carefully and clearly, in complete English sentences. This exam includes three pages of scratch paper at the end, which must be submitted, but will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page.

You are allowed to bring one letter-size single sided page of notes.

UC Berkeley Honor Code: *As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others.*

Sign here: _____

Question	Points
1	8
2	10
3	6
4	6
5	6
6	7
7	7
Total:	50

Do not turn over this page until your instructor tells you to do so.

Name and SID: _____

1. Circle true (**T**) or false (**F**) for each of the following. There is no need to provide an explanation.

   (a) (2 points) If E and F are events in a probability space such that $p(F) \neq 0$ then

   $$p(E|F) \leq p(E).$$

   T   F

   (b) (2 points) If E and F are independent events in a probability space, then E and $\overline{F}$ are also independent.

   T   F

   (c) (2 points) If $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$ are random variables such that $X(s) > Y(s)$ for all $s \in S$, then
   $$\mathbb{E}X > \mathbb{E}Y.$$

   T   F

   (d) (2 points) If X and Y are independent random variables then

   $$\mathbb{E}(X + Y)^2 = (\mathbb{E}X + \mathbb{E}Y)^2.$$

   T   F

[Scratch space below]

2. Nikhil is trying to solve a certain mathematical problem. There is a 2/3 chance that he will pass out while writing. If he passes out, there is a 3/4 chance that he will make an arithmetic mistake. If he doesn't pass out, there is only a 3/16 chance.

   (a) (3 points) What is the probability that Nikhil will make a mistake?

   (b) (3 points) Given that he does make a mistake, what is the probability that he passed out?

[Scratch Paper 1]

(c) (4 points) Suppose now that in an alternate universe, Nikhil passes out with probability $q \in (0, 1)$ instead of $2/3$, with everything else unchanged — i.e., we still have probability $3/4$ of a mistake if he does pass out, and $3/16$ of a mistake if he doesn't pass out.

Is there some value of q for which the events $E = \{\text{Nikhil makes a mistake}\}$ and $F = \{\text{Nikhil passes out}\}$ are independent? If so, find such a q, and if not prove that no such q exists.

3. (6 points) There are 5 distinguishable bins labeled $\{1, 2, 3, 4, 5\}$. How many ways are there of placing 100 indistinguishable balls into the bins, if each bin must have at least as many balls as its label? Be sure to explain your reasoning.

4. (6 points) Show using a combinatorial proof that:

$$\binom{2n}{3} = \binom{n}{3} + \binom{n}{3} + \binom{n}{2}n + n\binom{n}{2}.$$

[Scratch Paper 2]

5. (6 points) Show that any set of 76 distinct positive integers chosen from $\{1, \ldots, 100\}$ must contain 4 contiguous integers (i.e., $n, n+1, n+2, n+3$ for some n).

6. (7 points) Prove that for every positive integer $n \geq 1$, the sum

$$S_n = 1 + 3 + \ldots + (2n - 1)$$

is a perfect square.

7. (7 points) Consider the function recursively defined by:

$$f(1) = 3 \quad f(2) = 2 \quad f(3) = 1 \qquad f(k+1) = f(k) + f(k-1)f(k-2) \quad k \geq 3.$$

Prove that $f(n) \leq 2^{2^n}$ for all $n \geq 1$.

[Scratch Paper 3]