DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

Formulae

$$\int \tan(x)\,dx = \ln|\sec(x)| + C \qquad \int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$$

$$\int \frac{1}{1+x^2}dx = \arctan(x) + C \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$$

$$\frac{d\tan(x)}{dx} = \sec^2(x) \qquad \frac{d\sec(x)}{dx} = \tan(x)\sec(x)$$

$$1 = \sin^2(x) + \cos^2(x) \qquad 1 + \tan^2(x) = \sec^2(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$|E_{T,n}| \leq \frac{K(b-a)^3}{12n^2} \qquad |E_{S,n}| \leq \frac{K(b-a)^5}{180n^4}$$

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS FINISHED

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

Name and section: 1B Midterm 1 (002) Master Copy

GSI's name: ______________________

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This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Compute the following integrals:

(a) (10 points)

$$\int x\tan(x)\sec(x)\,dx$$

Solution:

$$\left.\begin{array}{l} f(x) = x \ , \ g'(x) = \tan(x)\sec(x) \\ f'(x) = 1 \ , \ g(x) = \sec(x) \end{array}\right\} \Rightarrow$$

$$\int x\tan(x)\sec(x)\,dx = x\sec(x) - \int \sec(x)\,dx$$

$$= x\sec(x) - \ln|\tan(x)+\sec(x)| + C$$

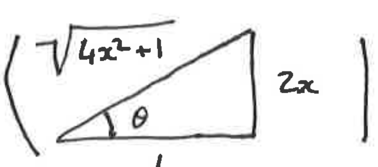
(b) (15 points)

$$\int_0^1 \frac{1}{(4x^2+1)^{3/2}}\,dx$$

Solution:

$$x = \frac{1}{2}\tan\theta \Rightarrow \frac{dx}{d\theta} = \frac{1}{2}\sec^2\theta \Rightarrow dx = \frac{1}{2}\sec^2\theta\,d\theta \Rightarrow$$

$$\int \frac{1}{(4x^2+1)^{3/2}}\,dx = \frac{1}{2}\int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{2}\int \cos\theta\,d\theta = \frac{1}{2}\sin\theta + C$$



$$= \frac{1}{2}\cdot\frac{2x}{\sqrt{4x^2+1}} + C = \frac{x}{\sqrt{4x^2+1}} + C$$

$$\Rightarrow \int_0^1 \frac{1}{(4x^2+1)^{3/2}}\,dx = \frac{1}{\sqrt{5}}$$

2. Determine if the following improper integrals are convergent or divergent. Justify your answers.

(a) (10 points)

$$\int_1^{\infty} \frac{|\sin(x) + \cos(x)|}{x^5 + 1} dx$$

Solution:

$0 \leq |\sin(x) + \cos(x)| \leq 1 + 1 = 2$ on $[1, \infty)$

$\Rightarrow 0 \leq \frac{|\sin(x) + \cos(x)|}{x^5 + 1} \leq \frac{2}{x^5 + 1} < \frac{2}{x^5}$ on $[1, \infty)$

$\int_1^{\infty} \frac{2}{x^5} dx$ convergent $\Rightarrow \int_1^{\infty} \frac{|\sin(x) + \cos(x)|}{x^5 + 1} dx$ convergent

(b) (15 points)

$$\int_0^{\infty} \frac{1}{(x-2)^3} dx$$

$\int_0^{\infty} \frac{1}{(x-2)^3} dx = \int_0^{2} \frac{1}{(x-2)^3} dx + \int_2^{3} \frac{1}{(x-2)^3} dx + \int_3^{\infty} \frac{1}{(x-2)^3} dx$

Let $u = x - 2 \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$

$\Rightarrow \int_2^{3} \frac{1}{(x-2)^3} dx = \int_0^{1} \frac{1}{u^3} du$

$\int_0^{1} \frac{1}{u^3} du$ divergent $\Rightarrow \int_2^{3} \frac{1}{(x-2)^3} dx$ divergent

$\Rightarrow \int_0^{\infty} \frac{1}{(x-2)^3} dx$ divergent

3. (a) (15 points) Express the following rational function

$$\frac{2x^2+x+1}{x^3+x}$$

as a sum of partial fractions.

Solution:

$$\frac{2x^2+x+1}{x^3+x} = \frac{2x^2+x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1) + (Bx+C)x}{x(x^2+1)} = \frac{(A+B)x^2 + Cx + A}{x(x^2+1}$$

$$\Rightarrow \begin{array}{l} A=1 \\ C=1 \\ B=1 \end{array} \Rightarrow \frac{2x^2+x+1}{x^3+x} = \frac{1}{x} + \frac{x+1}{x^2+1}$$

(b) (10 points) Hence evaluate the integral

$$\int \frac{2x^2+x+1}{x^3+x}\,dx$$

Solution:

$$\int \frac{2x^2+x+1}{x^3+x}\,dx = \int \frac{1}{x}\,dx + \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= \ln|x| + \frac{1}{2}\ln|x^2+1| + \arctan(x) + C$$

4. (25 points) Find the area of the surface of revolution of

$$y = \frac{e^{2x} + e^{-2x}}{4}$$

between $x = 0$ and $x = 1$.

Solution:

$$f(x) = \frac{e^{2x} + e^{-2x}}{4} \Rightarrow f'(x) = \frac{2e^{2x} - 2e^{-2x}}{4} = \frac{e^{2x} - e^{-2x}}{2}$$

$$\Rightarrow \quad 1 + f'(x)^2 = 1 + \frac{e^{4x}}{4} - \frac{1}{2} + \frac{e^{-4x}}{4}$$

$$= \frac{e^{4x}}{4} + \frac{1}{2} + \frac{e^{-4x}}{4} = \left(\frac{e^{2x} + e^{-2x}}{2}\right)^2$$

$$\Rightarrow \quad \text{Area} = \int_0^1 2\pi\left(\frac{e^{2x} + e^{-2x}}{4}\right) \cdot \frac{e^{2x} + e^{-2x}}{2}\, dx$$

$$= \frac{\pi}{4} \int_0^1 e^{4x} + 2 + e^{-4x}\, dx$$

$$= \frac{\pi}{4}\left(\frac{e^{4x}}{4} + 2x - \frac{e^{-4x}}{4}\right)\Big|_0^1$$

$$= \frac{\pi}{4}\left(\frac{e^4}{4} + 2 - \frac{e^{-4}}{4}\right)$$

5. (a) (15 points) For n a positive integer, let T_n be the trapezoidal approximation of the definite integral

$$\int_0^1 (x\sin(x) + 2\cos(x))dx.$$

How large do we need to choose n to be to guarantee that the estimate is within 0.01 of the true value? You do not need to give an exact answer, just a rough bound.

Solution:

$$f(x) = x\sin(x) + 2\cos(x) \Rightarrow f'(x) = \sin(x) + x\cos(x) - 2\sin(x)$$

$$\Rightarrow f''(x) = \cos(x) + \cos(x) - x\sin(x) - 2\cos(x) = -x\sin(x).$$

$|-x|, |\sin(x)| \leq 1$ on $[0,1] \Rightarrow |-x\sin(x)| \leq 1$ on $[0,1]$

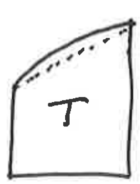
Choose $K = 1$. Need $n \geq 1$ such that

$$\frac{1 \cdot (1-0)^3}{12 \, n^2} = \frac{1}{12 n^2} \leq 0.01 \Rightarrow n \geq \sqrt{\frac{100}{12}}$$

(b) (10 points) Is this approximation an overestimate or an underestimate? Be sure to justify your answer.

Solution:

$f''(x) = -x\sin(x) \leq 0$ on $[0,1] \Rightarrow y = f(x)$ concave down on $[0,1] \Rightarrow T_n$ an underestimate



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