MATH 54 MIDTERM 2

April 5 2016 12:40-2:00pm

Your Name Student ID	SOLUTIONS & core skills expected to pass this course

Section number and leader	

Do not turn this page until you are instructed to do so.

No material other than simple writing utensils may be used. Show all your work in this exam booklet. There are blank pages in between the problems for scratch work.
If you want something on an extra page to be graded, label it by the problem number and write "XTRA" on the page of the actual problem.
In the event of an emergency or fire alarm leave your exam on your seat and meet with your GSI or professor outside.
If you need to use the restroom, leave your exam with a GSI while out of the room.

Point values are indicated in brackets to the left of each problem. Partial credit is given for explanations and documentation of your approach, even when you don't complete the calculation. In particular, if you recognize your result to be wrong (e.g. by checking!), stating this will yield extra credit.

When asked to explain/show/prove, you should make clear and unambiguous statements, using a combination of formulas and words or arrows. Graders will disregard formulas whose meaning is unclear.

[5] **1a)** Find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1, (1+t)^2, (1-t)^2, t^3\}$ of $\mathbb{P}_3$ to the standard basis $\mathcal{C} = \{1, t, t^2, t^3\}$.

$$[1]_e = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

core skill

$$[(1\pm t)^2]_e = [1 \pm 2t + t^2]_e = \begin{bmatrix}1\\ \pm 2\\ 1\\ 0\end{bmatrix}$$

$$[t^3]_e = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$\Rightarrow \underset{e \leftarrow \mathcal{B}}{P} = \begin{bmatrix}1 & 1 & 1 & 0\\ 0 & 2 & -2 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Note: Trying to calculate the inverse is bad test taking strategy. Once it gets calculation intensive, you should ask yourself whether you misread the question.

[7] **1b)** Let V be a finite dimensional vector space. State the definition for the dimension of a subspace H. Then explain which subspaces of V have $\dim H = \dim V$.

- $\dim H$ = number of vectors in a basis of H
- $\dim H = \dim V$ only for $H = V$
- Explanation: $\{b_1 .. b_n\}$ basis of H, $\dim V = n$

$\Rightarrow \{b_1 .. b_n\}$ lin. indep. in V

$\Rightarrow \{b_1 .. b_n\}$ basis of V $\Rightarrow$ $H = V$

general fact: n lin. indep. vectors in an n-dim space also span $\Rightarrow$ are a basis

Note: You should know these to pass the course.

Whereas this was the A+ question

[8] **1c)** State the definition of $\mathcal{B} = \{b_1, \ldots, b_n\}$ being a basis of a vector space V.
Then use the properties in this definition to explain why the linear transformation $T : \mathbb{R}^n \to V$ given by $T(\mathbf{x}) = x_1 b_1 + \ldots + x_n b_n$ is an isomorphism.

$\mathcal{B}$ is a basis if
- $V = \text{span}\{b_1 .. b_n\}$
- $b_1 .. b_n$ lin. indep.

Note: You should know these to pass this course.

T is an isomorphism if onto and one-to-one

- onto: given v in V, can find $\underline{c}$ with $T(\underline{c}) = v$

$$T(\underline{c}) = c_1 b_1 + .. + c_n b_n$$

by definition of $\mathcal{B}$ spanning V

- one-to-one: since T is linear, suffices to check "$T(\underline{c}) = 0 \Rightarrow \underline{c} = \underline{0}$"

$$T(\underline{c}) = c_1 b_1 + .. + c_n b_n = 0 \implies c_1 = .. = c_n = 0$$

by definition of $b_1 .. b_n$ lin. indep.

[6] **2a)** Let A and B be 3×3 matrices with $\det(A) = -1$ and $\det(B) = 3$. State appropriate properties of determinants to compute the following:

$\det(2A) = \ldots 2^3 \det A = -8$

$\det(cA) = \underbrace{\det(cI_n)}_{= c^n \det I_n = c^n} \det A$

Note: First two parts are core knowledge for this course.

$\det(BAB^T) = \ldots \det B \det A \underbrace{\det(B^T)}_{= \det B} = 3 \cdot (-1) \cdot 3 = -9$

$\det(CD) = \det C \cdot \det D$

The volume of the parallelepiped spanned by the columns of the matrix $B^{-1}A$ is ...

$|\det(B^{-1}A)| = |\det(B^{-1}) \det A| = \left| \frac{1}{\det B} \det A \right| = \left| \frac{-1}{3} \right| = \frac{1}{3}$

$\det(B^{-1}) = \frac{1}{\det B}$

[8] **2b)** Find (possibly complex) eigenvectors for each eigenvalue of $A = \begin{bmatrix} 5 & 0 & -5 \\ 0 & 5 & 0 \\ 5 & 0 & 5 \end{bmatrix}$.

(Hint: When calculating the characteristic polynomial, note a common factor that you should not multiply out. Then finding the roots only requires solving a quadratic equation.)

$$\det(A-\lambda I) = (5-\lambda)^3 - (-5)(5-\lambda)5$$
$$= (5-\lambda)(\lambda^2 - 10\lambda + 50)$$

roots : $5, \quad \frac{10}{2} \pm \sqrt{5^2 - 50} = 5 \pm 5i$

Note: Getting this right, even when stressed, is a core skill for this course.

$\lambda = 5$ eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ by inspection

$\lambda = 5-5i$ eigenvector :

$$\mathrm{Nul}(A-\lambda I) = \mathrm{Nul}\begin{bmatrix} 5i & 0 & -5 \\ 0 & 5i & 0 \\ 5 & 0 & 5i \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} i & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rightsquigarrow$ any complex multiple of $\begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}$

$\lambda = 5+5i$ eigenvector $\begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix}$ by complex conjugation

[6] **2c)** Suppose that A is a 4×4 matrix with characteristic polynomial $\lambda(\lambda + \sqrt{5})(\lambda - \sqrt{7})^2$, and that $\mathbb{R}^4$ has a basis which consists of eigenvectors of A. Specify a diagonal matrix D that A is similar to, state this similarity as a formula involving A, D, and another matrix P, and explain how to find P.

$$P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sqrt{5} & 0 & 0 \\ 0 & 0 & \sqrt{7} & 0 \\ 0 & 0 & 0 & \sqrt{7} \end{bmatrix}$$

Note: knowing D from given info is core skill for passing this course.

$$P = \begin{bmatrix} \text{eigenvector for } 0, & \text{eigenvector for } -\sqrt{5}, & \text{eigenvector for } \sqrt{7}, & \text{eigenvector for } \sqrt{7} \end{bmatrix}$$

lin. indep.

Alternative:
different order of diagonal entries and eigenvectors

[6] **3a)** Find the general solution of $y^{(7)} + 4y^{(5)} - 3y^{(3)} - 18y' = 0$.
Then give an example of initial conditions that would specify a unique, nonzero solution.
You may use the identity $r^7 + 4r^5 - 3r^3 - 18r = r(r^2-2)(r^2+3)^2$.

core skills and Knowledge

roots: $0, \ \pm\sqrt{2}, \ \underbrace{\pm\sqrt{3}\,i}_{\text{double}}$

$$y(t) = C_1 + C_2 e^{\sqrt{2}t} + C_3 e^{-\sqrt{2}t} + C_4 \cos\sqrt{3}t + C_5 \sin\sqrt{3}t + C_6 t\cos\sqrt{3}t + C_7 t \sin\sqrt{3}t$$

initial conditions e.g. at $t=0$

$$\begin{bmatrix} y(0) \\ y'(0) \\ \vdots \\ y^{(6)}(0) \end{bmatrix} = \text{some fixed vector} \neq \underline{0} \text{ in } \mathbb{R}^7$$

[8] **3b)** Find the solution of $y'' + y' = t^2$ with $y(0) = 0$ and $y'(0) = 0$.

aux. eq.: $r^2 + r = 0$

roots: $r = 0, -1$ $\longrightarrow$ "resonant" Ansatz needed since t^2 has "frequency" $r=0$

$t^2 = t^2 e^{rt}, r = 0$

$\Downarrow$

general homog. sol. $c_1 + c_2 e^{-t}$ || core skill

Ansatz for particular sol.: $y(t) = at + bt^2 + ct^3$

plug in $y' = a + 2bt + 3ct^2$

$y'' = 2b + 6ct$

|| core skill: making some Ansatz and coming e.g. to "doesn't work" conclusion

$$t^2 = y'' + y' = (a+2b) + (2b+6c)t + 3ct^2$$

solve

$$\Longleftrightarrow \quad \begin{aligned} 1 &= 3c \\ 0 &= 2b + 6c \\ 0 &= a + 2b \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} c &= 1/3 \\ b &= -3c = -1 \\ a &= -2b = 2 \end{aligned}$$

plug back

general solution: $y(t) = \frac{1}{3}t^3 - t^2 + 2t + c_1 + c_2 e^{-t}$

initial conditions: $y(0) = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2 \Rightarrow c_1 = -2$

$y'(0) = 2 - c_2 = 0 \Rightarrow c_2 = 2$

$$\Rightarrow \underline{\underline{y(t) = \frac{1}{3}t^3 - t^2 + 2t - 2 + 2e^{-t}}}$$

[6] **3c)** Find the general solution of $L[y] = \frac{1}{9}e^{-t} + 1$ and explain why there cannot be any other solutions, using only definitions and the following information (no theorems etc.):

1.) $L : \mathcal{C}^\infty \to \mathcal{C}^\infty$ is a linear transformation,
2.) kernel $L = \text{span}\{e^{-t}, \cos 3t, \sin 3t\}$,
3.) $L[te^{-t}] = e^{-t} + 9$.

$$L[y] = \tfrac{1}{9}e^{-t} + 1$$

$$\overset{(3)}{\Longleftrightarrow} \quad L[y] = \tfrac{1}{9}L[te^{-t}]$$

$$\overset{(1)}{\Longleftrightarrow} \quad L[y - \tfrac{1}{9}te^{-t}] = 0$$

$$\overset{\text{DEF}}{\Longleftrightarrow} \quad y - \tfrac{1}{9}te^{-t} \text{ in kernel } L$$

$$\overset{(2)}{\Longleftrightarrow} \quad y - \tfrac{1}{9}te^{-t} \text{ in span}\{e^{-t}, \cos 3t, \sin 3t\}$$

Since these are all "true if and only if", the solutions are exactly all (and no others) of the form

$$y(t) = \tfrac{1}{9}te^{-t} + c_1 e^{-t} + c_2 \cos 3t + c_3 \sin 3t$$

Core skill: reading this off from given info

[6] **4a)** Rewrite the ODE $y^{(4)} + 3y' + 5y = \cos 2t$ into a first order system for a vector function. Then assume that a solution of the system is given and explain how to obtain a solution y of the ODE.

with $\underline{x}(t) = \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \\ y'''(t) \end{bmatrix}$ $\qquad \underline{x}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -3 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos 2t \end{bmatrix}$

solution y is first component of system solution $\underline{x}$

Core skill: knowing basic substitution and shape of system at least roughly

[7] **4b)** Find the solution of $\mathbf{x}' = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

$\det(A-\lambda I) = \lambda^2+1 \longrightarrow$ eigenvalues $\pm i$

i-eigenvector: $\mathrm{Nul}(A-iI) = \mathrm{Nul}\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix} = \mathrm{span}\begin{bmatrix} i \\ 1 \end{bmatrix}$

$\Rightarrow$ complex solution $e^{it}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} i\cos t - \sin t \\ \cos t + i\sin t \end{bmatrix} = \underline{z}(t)$

(or any complex multiple thereof)

$\Rightarrow$ real solutions $\underline{x}_1(t) = \mathrm{Re}\, \underline{z}(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$

$\underline{x}_2(t) = \mathrm{Im}\, \underline{z}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$

Core skills

$\Rightarrow$ general solution $\underline{x}(t) = c_1 \underline{x}_1(t) + c_2 \underline{x}_2(t)$

initial value: $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \underbrace{\underline{x}_1(0)}_{= \begin{bmatrix} 0 \\ 1 \end{bmatrix}} + c_2 \underbrace{\underline{x}_2(0)}_{= \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$\Rightarrow c_1 = 1, \; c_2 = 0$

$\Rightarrow$ solution $\underline{x}(t) = \underline{x}_1(t) = \underline{\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}}$

[7] **4c)** The system $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \mathbf{x}$ has a solution $\mathbf{x}_1(t) = e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find a linearly independent solution by using the Ansatz ("educated guess") $\mathbf{x}(t) = te^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{2t}\mathbf{v}$ for an unknown vector $\mathbf{v}$ in $\mathbb{R}^2$. Document your steps (plug in, solve, plug back).

<u>plug in:</u> $\underline{x}' = e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2te^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^{2t}\underline{v}$

$\overset{?}{\|}$

$A\underline{x} = te^{2t}\underbrace{A\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} + e^{2t}A\underline{v}$

core skill

<u>solve:</u> $\Longleftrightarrow \quad e^{2t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\underline{v}\right) = e^{2t}A\underline{v}$

$\Longleftrightarrow \quad \underbrace{(A-2I)}_{\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}}\underline{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Longleftrightarrow \quad \underline{v} = \begin{bmatrix} x \\ 1/3 \end{bmatrix}$ any x

<u>plug back:</u> $\underline{x}(t) = te^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{2t}\begin{bmatrix} x \\ 1/3 \end{bmatrix}$ any x (e.g. pick $x=0$)

Alternative calculation: $\underline{x}(t) = t\,\underline{x}_1(t) + e^{2t}\underline{v}$; $\underline{x}_1' = A\underline{x}_1$ (*)

plug in: $\underline{x}' = \underline{x}_1 + t\underline{x}_1' + 2e^{2t}\underline{v}$

$\overset{?}{\|}$ $\qquad /\!/ (*)$

$A\underline{x} = A\underline{x}_1 + e^{2t}A\underline{v}$

$\Longleftrightarrow \quad \underline{x}_1 + 2e^{2t}\underline{v} \;\|\; e^{2t}A\underline{v} \quad \Longleftrightarrow$ as above