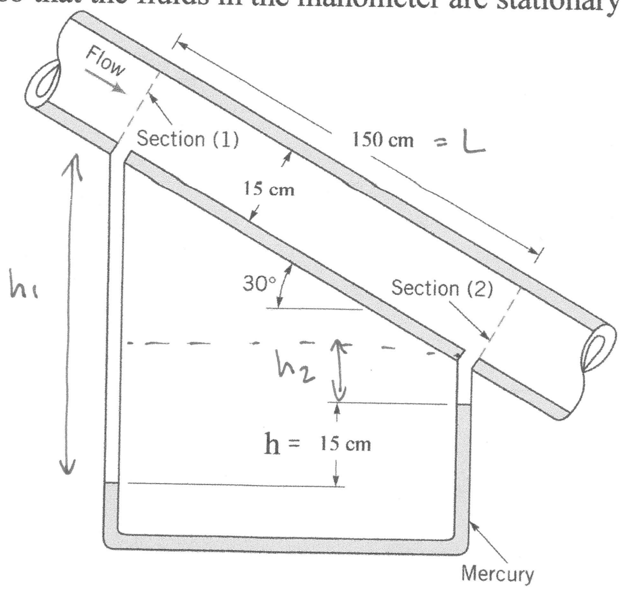
1) Viscous pipe flow (25 points)

Water flows through an inclined pipe. A U-tube manometer with fluid density ρ_m = 13,600 kg/m^3 measures the pressure difference between points 1 and 2 as shown. The pipe flow is steady so that the fluids in the manometer are stationary.



a) Find the pressure difference (P_1-P_2) between sections 1 and 2. (5 points)

Follow manometer

$$P_1 + \gamma h_1 - \gamma_m h - \gamma h_2 = P_2$$

$$P_1 - P_2 = \gamma_m h + \gamma h_2 - \gamma h_1 \qquad h_1 - h - h_2 = L \sin 30 = 0.5L$$

$$P_1 - P_2 = \gamma_m h - \gamma(0.5L) - \gamma h \qquad h_1 - h_2 = 0.5L + h$$

$$\gamma_m = \rho_m g$$
$$\gamma = \rho_{water} g$$

$$P_1 - P_2 = 11.18 \text{ kPa}$$

b) Find the head loss (h_L) between sections 1 and 2. (5 points)

$$\frac{P_1}{\gamma} + \cancel{\frac{V_1^2}{2g}} + z_1 = \frac{P_2}{\gamma} + \cancel{\frac{V_2^2}{2g}} + z_2 + h_L$$

velocities same by mass conservation

$$z_1 - z_2 = L \sin 30 = 0.5 L$$

$$P_1 - P_2 = \gamma(z_2 - z_1) + \gamma h_L$$

$$\underbrace{\gamma_m h - \gamma(h + 0.5L)} = -\gamma(0.5L) + \gamma h_L$$

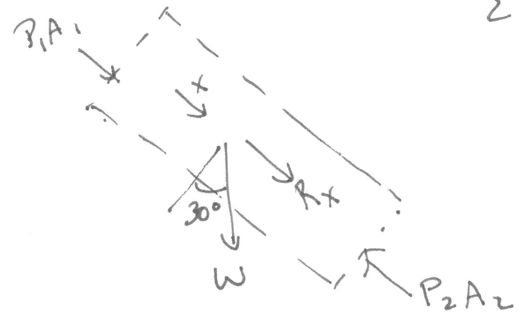
11.18 kPa from (a)

$$h_L = \frac{\gamma_m h - \gamma(h+0.5L) + \gamma(0.5L)}{\gamma} = \underline{1.89\ m}$$

c) Find the net axial force exerted by the pipe wall on the flowing water between sections 1 and 2. (10 points)

Apply momentum equation in axial direction

$$\Sigma F_x = \underbrace{\frac{\partial}{\partial t}\int_{cv} \rho v \, d\forall}_{0 \text{ steady}} + \int_{cs} \rho v \vec{v}\cdot\hat{n}\, dA$$



$$P_1A_1 - P_2A_2 + R_x + W \sin 30 = \underbrace{\dot{m}V_{out} - \dot{m}V_{in}}_{=0 \text{ since } V_{in} = V_{out}}$$

$$\underbrace{(P_1 - P_2)}_{\text{from (a)}} A + W \sin 30 = -R_x$$

$$A = \frac{\pi D^2}{4} = 0.0177\ m^2 \qquad W = \rho g A L = 260\ N$$

$$D = 0.15\ m$$

$$R_x = -W \sin 30 - (P_1 - P_2) A$$

$$R_x = \underline{-327.6\ N}$$

this is force of pipe on water and points opposite flow direction

d) What value would the manometer show for h if there were no flow in the pipe? (5 points)

Hydrostatics → no flow

Follow manometer:

$$P_1 + \gamma h_1 - \gamma_m h - h_2 \gamma = P_2$$

In pipe it's hydrostatic, so

$$P_2 = P_1 + \gamma (z_1 - z_2)$$

$$\therefore \quad \cancel{P_1} + \gamma h_1 - \gamma_m h - h_2 \gamma = \cancel{P_1} + \gamma (0.5L)$$

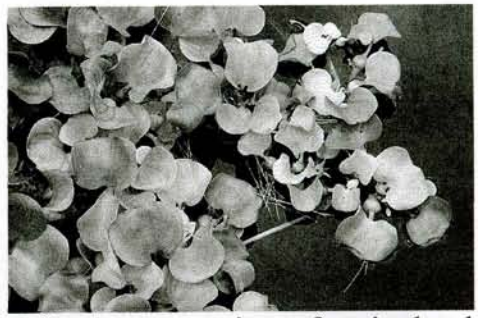
$$\gamma (h_1 - h_2) - \gamma_m h = \gamma (0.5L)$$

$$\gamma \cancel{(0.5L)} - \gamma_m h = \gamma \cancel{(0.5L)}$$

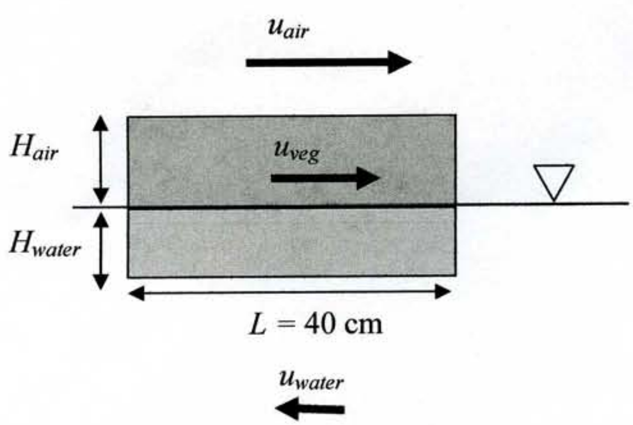
$$\underline{\underline{h = 0}}$$

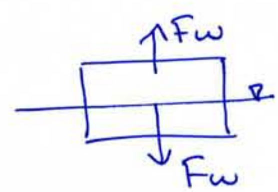
2) Drag force on vegetation raft (25 points)

Water hyacinth is a floating aquatic plant which tends to clump together to form vegetation rafts, as illustrated below.



a) Assuming we can conceptualize the floating vegetation raft as in the sketch below, determine the depth to which the plant is submerged (H_{water}). The raft has an average density of $\rho_f = 400$ kg/m^3 and the total height is $H = H_{air} + H_{water} = 25$ cm. Width into the page is $b = 10$ cm. (6 points)





Floating raft → weight balances buoyancy

$$F_B = F_W$$

$$\gamma_{water} \forall_{displaced} = \gamma_{raft} \forall_{raft}$$

$$\rho_w g H_{water} b = \rho_f g H b$$

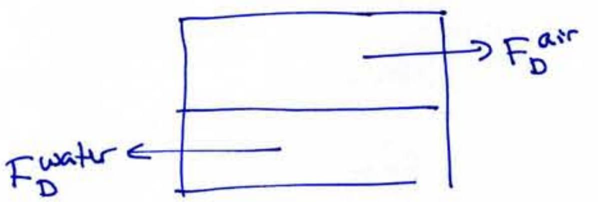
$$H_{water} = \frac{\rho_f}{\rho_w} H = \frac{400 \text{ kg/m}^3}{1000 \text{ kg/m}^3} H = 0.4 (25 \text{ cm}) = \underline{\underline{10 \text{ cm}}}$$

b) The vegetation raft experiences drag due to the wind and due to the water. The combined forces cause the raft to move at a velocity of u_{veg}. The drag due to each fluid can be found by using the relative velocities as

$$F_D^{air} = \frac{1}{2}\rho_{air} C_D^{air} A_{air}\left(u_{air} - u_{veg}\right)^2$$

$$F_D^{water} = \frac{1}{2}\rho_{water} C_D^{water} A_{water}\left(u_{water} - u_{veg}\right)^2$$

where A is the frontal area given by $A = Hb$ in each fluid, where $b = 10$ cm. Set up a force balance to find the vegetation raft velocity under steady state given $u_{air} = 10$ m/s, $u_{water} = -0.5$ m/s, $C_D^{air} = 1.0$, $C_D^{water} = 2.0$. (10 points)



at steady state $\Sigma F = 0$

$$F_D^{air} = F_D^{water}$$

$$\tfrac{1}{2}\rho_a C_D^a A^a (u_a - u_v)^2 = \tfrac{1}{2}\rho_w C_D^w A^w (u_w - u_v)^2$$

$$\frac{\rho_a C_D^a H^a b}{\rho_w C_D^w H^w b} = \frac{(u_w - u_v)^2}{(u_a - u_v)^2}$$

$$\underbrace{\pm\sqrt{\frac{\rho_a C_D^a H^a}{\rho_w C_D^w H^w}}}_{\gamma} = \frac{u_w - u_v}{u_a - u_v}$$

$\gamma = 0.03$

for positive root: $\gamma(u_a - u_v) = u_w - u_v$

$$u_v - \gamma u_v = u_w - \gamma u_a$$

$$u_v = \frac{u_w - \gamma u_a}{1 - \gamma} = \underline{-0.19 \text{ m/s}}$$

for negative root: $-\gamma(u_a - u_v) = u_w - u_v$

$$-\gamma u_a - u_w = -\gamma u_v - u_v$$

$$u_v = \frac{\gamma u_a + u_w}{1 + \gamma} = -0.82 \text{ m/s}$$ ← this is unphysical because it can't go faster than water velocity

c) Given a water velocity of $u_{water} = -0.5$ m/s, find the wind velocity u_{air} required to have $u_{veg} = 0$. (3 points)

Use $u_v = \dfrac{u_w - \gamma u_a}{1-\gamma}$ and set $u_v = 0$

$\rightarrow u_w = \gamma u_a$

$u_a = \frac{1}{\gamma} u_w = -16.7$ m/s ✗ this is unphysical use other root

$\Rightarrow u_v = \dfrac{u_w + \gamma u_a}{1+\gamma}$ and set $u_v = 0$

$u_w = -\gamma u_a = \underline{\underline{16.7 \text{ m/s}}}$ makes sense that wind and water are opposite directions to give $u_{veg} = 0$

d) If the size of the raft becomes longer (L increases), do you expect friction drag to increase or decrease compared to form drag (also known as pressure drag)? How could you incorporate the new length L in your drag formula given that $A = Hb$ and does not depend on L? (6 points)

As L increases friction drag will increase compared to form drag because total shear stresses will increase due to longer length.

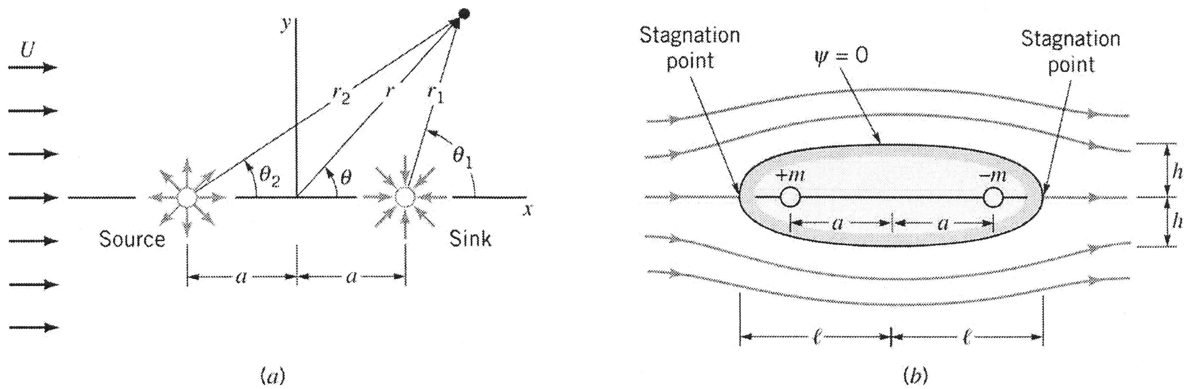
The current formula uses a C_D that is based on frontal area therefore to incorporate L effects we have to use a new value of C_D.

3) Groundwater pumping – potential flow (20 points)

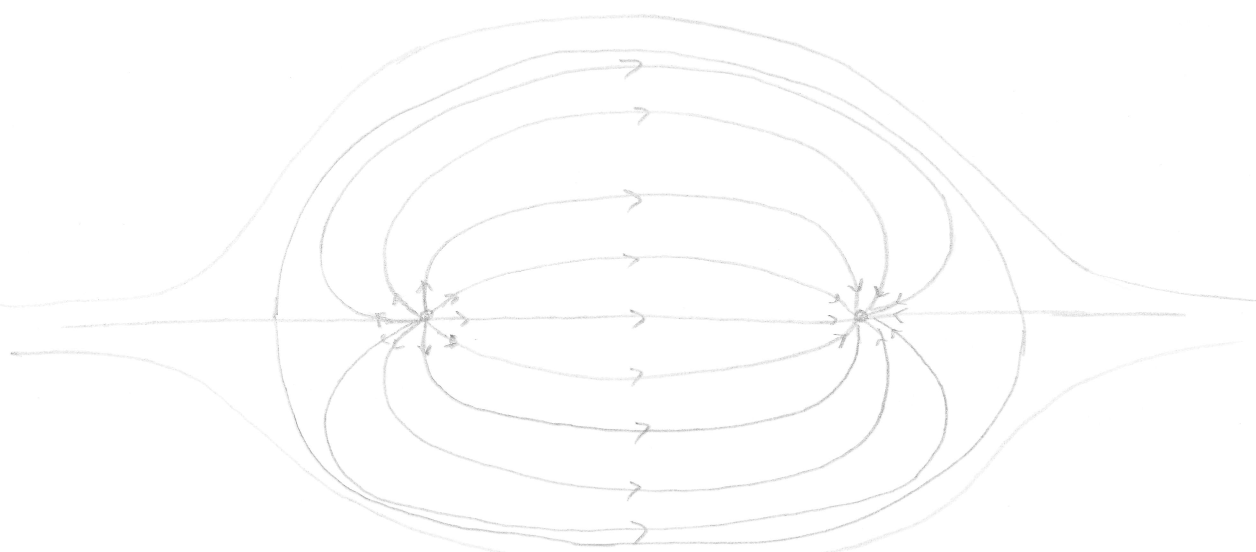
Toxic chemical spills can cause groundwater contamination in the subsurface. A common treatment approach is to inject clean water into the ground and pump dirty water out, known as "pump and treat". This creates a capture zone in the flow that can be described by a Rankine oval. Thus any contaminated water inside the oval can be pumped out and will no longer flow downstream.

The streamfunction for flow around the Rankine oval is given by superposition of uniform flow and a source/sink pair:

$$\psi = Ur\sin\theta - \frac{m}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$$



a) Sketch streamlines inside the oval (use a new sketch below). (4 points)



b) Find expressions for the velocity components v_r and v_θ. (5 points)

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad V_\theta = -\frac{\partial \psi}{\partial r} \qquad \text{use } \frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$$

$$V_r = \frac{1}{r}\left[U r \cos\theta - \frac{m}{2\pi}\frac{\frac{2ar}{r^2-a^2}\cos\theta}{1+\left(\frac{2ar\sin\theta}{r^2-a^2}\right)^2}\right]$$

$$V_r = U\cos\theta - \frac{m}{2\pi}\frac{\frac{2a\cos\theta}{r^2-a^2}}{1+\left(\frac{2ar\sin\theta}{r^2-a^2}\right)^2}$$

$$V_\theta = -U\sin\theta + \frac{m}{2\pi}\frac{\left((r^2-a^2)(2a\sin\theta) - 2ar\sin\theta\,(2r)\right)/(r^2-a^2)^2}{1+\left(\frac{2ar\sin\theta}{r^2-a^2}\right)^2}$$

$$V_\theta = -U\sin\theta + \frac{m}{2\pi}\frac{(r^2-a^2)(2a\sin\theta) - 4ar^2\sin\theta}{(r^2-a^2)^2 + (2ar\sin\theta)^2}$$

c) How would you approximate the derivative you needed above for v_θ numerically using a central finite difference scheme? (3 points)

$$V_\theta = -\frac{\partial \psi}{\partial r} \approx -\frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{2\Delta r} + O(\Delta r^2)$$

where i is index for r

j is index for θ

d) What is the magnitude of the velocity at $r = 0$? (3 points)

at $r=0$ $\quad V_r = U\cos\theta + \frac{m}{\pi}\frac{\cos\theta}{a} = \left(U + \frac{m}{\pi a}\right)\cos\theta$

$V_\theta = -U\sin\theta - \frac{m}{\pi}\frac{\sin\theta}{a} = -\left(U + \frac{m}{\pi a}\right)\sin\theta$

$$|V| = \sqrt{V_r^2 + V_\theta^2} = \sqrt{\left(U + \frac{m}{\pi a}\right)^2\cos^2\theta + \left(U + \frac{m}{\pi a}\right)\sin^2\theta}$$

$$|V| = U + \frac{m}{\pi a}$$

e) Find an expression for the pressure at $r=0$ using p_0 as a reference at an arbitrary point far away. (5 points)

$$P_0 + \frac{1}{2}\rho U^2 + \cancel{z_0} = P + \frac{1}{2}\rho V^2 + z$$

neglect

at $r=0$

$$P(r=0) = P_0 + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho V^2$$

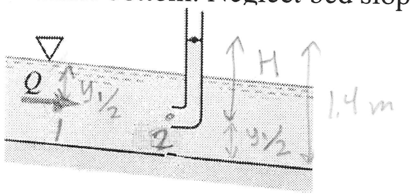
$$P(r=0) = P_0 + \frac{1}{2}\rho\left[U^2 - \left(U + \frac{m}{a\pi}\right)^2\right]$$

from part d

$$P(r=0) = P_0 - \frac{1}{2}\rho\left(\frac{m^2}{a^2\pi^2} + 2\frac{mU}{a\pi}\right)$$

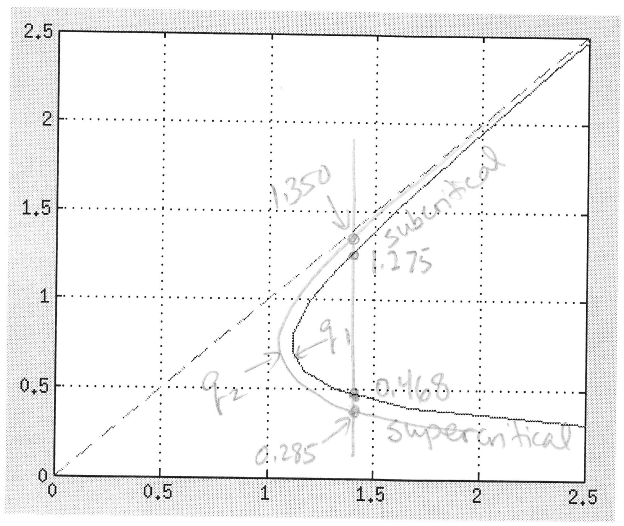
4) Channel design (30 points)

Water flows in a rectangular channel of width $b_1 = 2$ m at a rate of $q = 2\ m^2/s$. When a Pitot tube is placed in the stream, water in the tube rises to a level of 1.4 m above the channel bottom. Neglect bed slope and friction.



← Assume pitot tube is at depth $y_1/2$ (works for other choices too)

a) Write the equations you need to solve to determine V_1 and y_1 for the conditions shown. Modify them to produce a single cubic equation for the two possible flow depths in the channel but do not solve the equation. The roots to your cubic equation are (1.275, 0.468, -0.342). Label and describe the two possible solutions on the specific energy diagram below. (10 points)



Apply Bernoulli from 1 to 2

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + \cancel{z_1} = \frac{P_2}{\gamma} + \cancel{\frac{V_2^2}{2g}} + \cancel{z_2}$$

straight streamlines open channel hydrostatic

$V_2 = 0$ stagnation point

$P_1 = \gamma(y_1/2)$

$P_2 = \gamma H$ (H is Pitot tube height)

$q = y_1 V_1 \rightarrow \boxed{V_1 = q/y_1}$

get V_1 after y_1 is found

$$\frac{y_1}{2} + \frac{q^2}{2g y_1^2} = H \qquad H + y_1/2 = 1.4$$

$$y_1 + \frac{q^2}{2g y_1^2} = H + y_1/2 = \boxed{1.4\ m = E_1}$$

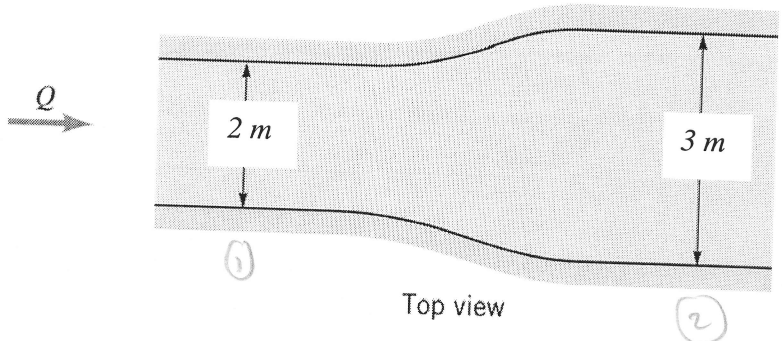
cubic eqn: $\boxed{y_1^3 - y_1^2 E_1 + \frac{q^2}{2g} = 0}$

3 roots (1.275, 0.468, −0.342)

subcritical, supercritical, unphysical

2 alternate depths possible

b) Suppose the channel now transitions from a width of $b_1 = 2$ m to $b_2 = 3$ m. Calculate the new alternate depths are and show which one will be chosen depending on the choice of upstream depth (i.e. you should have two answers and two explanations). The roots to your new cubic equation are (1.350, 0.285, -0.235). Label these on your diagram above. (10 points)



Top view

$E_1 = E_2 = \underline{1.4\ m}$

$$y_1 + \frac{V_1^2}{2g} = y_2 + \frac{V_2^2}{2g}$$

$Q_1 = Q_2$

$q_1 = y_1 V_1 \qquad q_2 = y_2 V_2$

$b_1 q_1 = b_2 q_2$

$$y_1 + \frac{q_1^2}{2g y_1^2} = y_2 + \frac{q_2^2}{2g y_2^2}$$

$$q_2 = \frac{q_1 b_1}{b_2}$$

$$E_1 = y_2 + \frac{q_2^2}{2g y_2^2}$$

$$q_2 = 2\,\frac{m^2}{s} \cdot \frac{2\ m}{3\ m} = \underline{\frac{4}{3}\,\frac{m^2}{s}}$$

$q_2 < q_1$ so move to lower q curve on energy diagram

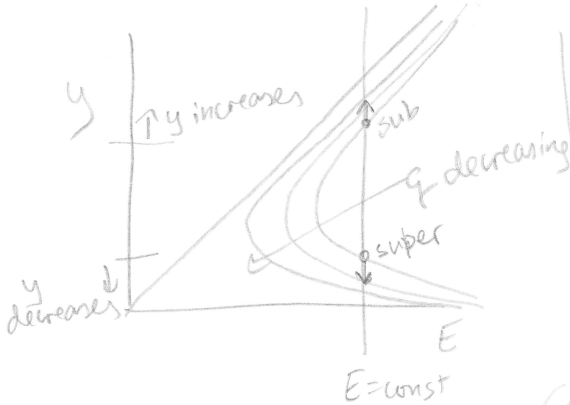
new cubic to solve $\boxed{y_2^3 - E_1 y_2^2 + \frac{q_2^2}{2g} = 0}$

3 roots (1.350, 0.285, -0.235)
sub super unphysical

If y_1 is subcritical, y_2 stays subcritical (1.35 m)
" " " supercritical, " " supercritical (0.285 m)

→ stay on same energy line but move to new q curve

c) What will happen to the velocity in the channel if it gets wider and wider? Explain your reasoning on the energy diagram. (5 points)



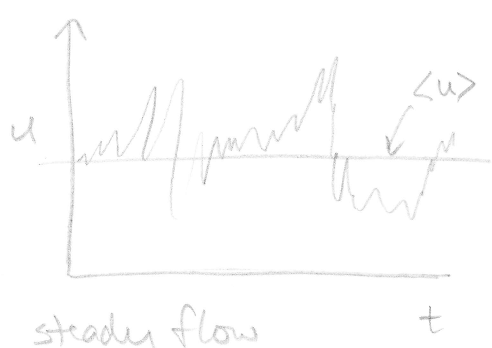
Channel width changes so move to new q curve but stay on same energy line

If b increases (wider channel), q decreases.

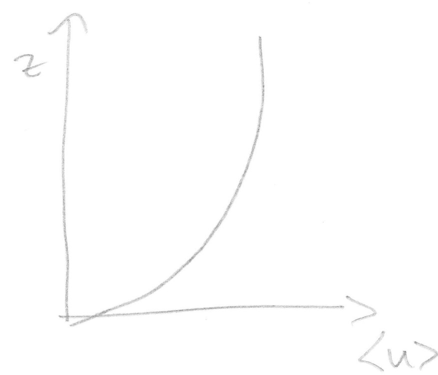
Thus depth becomes even deeper for subcritical (velocity decreases) and shallower for supercritical (velocity increases).

∴ q ↓ means y → E for subcritical
q ↓ " y → 0 " supercritical

d) Sketch what a velocity probe (like the acoustic doppler velocimeter (ADV) which you used in lab 4) would record if it were placed in the channel. Plot u vs. time. Also plot <u> vs. depth where < > denotes a time average. Label/describe your plots. (5 points)



steady flow on average <u> but turbulent



<u> decreases towards the bottom of the channel to zero on the wall (no slip / friction effects). Logarithmic profile