Physics 7C, Fall 2015
Midterm 2 Solutions

Problem 1

At $x = L$, we observe light of wavelength λ to be particularly bright, which means we are observing constructive interference at that wavelength. Figure 1 shows the two interfering light rays that we need to consider. The angular opening θ of the sliver of glass is very small, so the incoming and outgoing rays are all on top of each other and perpendicular to the x axis, although they have been drawn at incoming and outgoing angles in Figure 1 to make them distinguishable. Outside of the glass, the index of refraction is $n_{\text{air}} \approx 1$. The glass has index of refraction $n > n_{\text{air}}$. At $x = L$, the thickness of the glass is $d = L \tan\theta$.

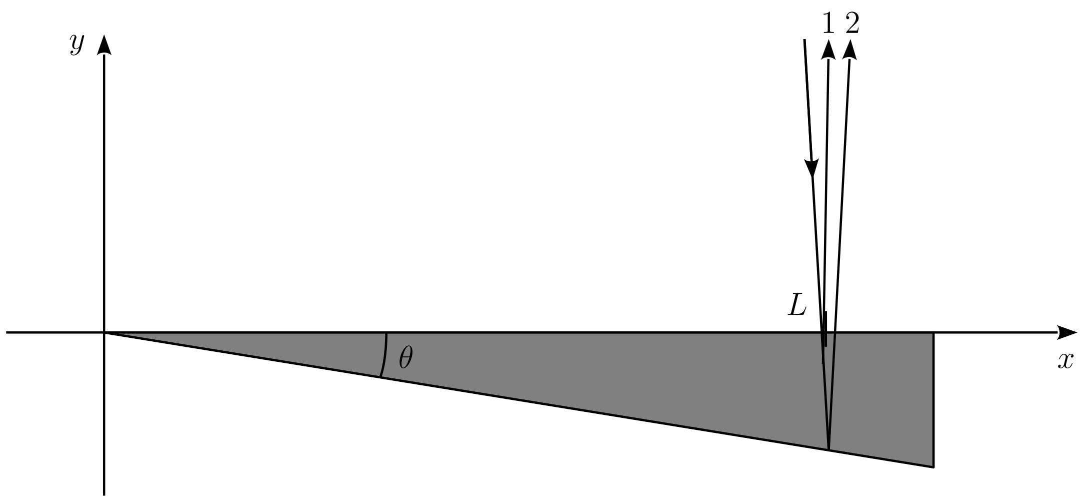

Figure 1

Ray 1:
Ray 1 accumulates a phase of π upon reflection from the upper glass/air interface, since $n > n_{\text{air}}$. Therefore, it accumulates a phase of $\phi_1 = \pi$.

Ray 2:
Ray 2 travels a distance $2d$ in the glass. For light of wavelength λ in air, which we observe to be bright, ray 2 accumulates $\frac{2d}{\lambda_n}$ wavelengths as it travels through the glass, where $\lambda_n = \frac{\lambda}{n}$ is the wavelength of the light in the glass. Thus, ray 2 accumulates a phase of $2\pi\frac{2d}{\lambda_n} = 2\pi\frac{2nL\tan\theta}{\lambda}$ traveling through the glass. Ray 2 does not accumulate a phase of π upon reflection from the lower air/glass interface, since $n_{\text{air}} < n$. Therefore, ray 2 accumulates a

phase of $\phi_2 = 2\pi\frac{2nL\tan\theta}{\lambda}$.

Phase difference:
The phase difference between the two rays is

$$\Delta\phi = \phi_2 - \phi_1 = 2\pi\frac{2nL\tan\theta}{\lambda} - \pi \tag{1}$$

The phase difference must be an integer multiple of 2π to achieve constructive interference at wavelength λ, so $\Delta\phi = 2\pi m$, where m is an integer:

$$\begin{aligned} 2\pi m &= \Delta\phi & (2)\\ &= 2\pi\frac{2nL\tan\theta}{\lambda} - \pi & (3)\\ m &= \frac{2nL\tan\theta}{\lambda} - \frac{1}{2} & (4)\\ \tan\theta &= \frac{2m+1}{4}\frac{\lambda}{nL} & (5) \end{aligned}$$

At $x = L$, we observe no other wavelengths accentuated, so the thickness $d = L\tan\theta$ must be the minimum thickness for constructive interference at wavelength λ, which implies $m = 0$. Therefore,

$$\begin{aligned} \tan\theta &= \frac{1}{4}\frac{\lambda}{nL} & (6)\\ \theta &= \tan^{-1}\left(\frac{1}{4}\frac{\lambda}{nL}\right) & (7) \end{aligned}$$

Physics 7C, Fall 2015, Midterm 2, Problem 2

At $t = 0$ in his stationary frame S, Tom suddenly ***sees*** Dick throw a ball towards him with velocity $-u_B = -\frac{3}{5}c$. Tom catches the ball with his robotic arm at a time T. Harry is in the moving frame S', moving to the right with speed $v = \frac{4}{5}c$. The origins of S and S' are at the same point at $t' = t = 0$.

a. Where, x_0, and when, t_0, was the ball thrown in Tom's frame?

There are three events to consider: Dick throws ball ($t_D = t_0, x_D = x_0$), Tom sees Dick throw ball ($t_S = 0, x_S = 0$), and Tom catches ball ($t_C = T, x_C = 0$). *We must take into account the time it takes for light to travel between Dick and Tom.*

The time T between Tom seeing and catching can be used to first solve for x_0. We know that the time $t_S - t_D$ for light to reach Tom from Dick is $\frac{x_0}{c}$. The time $t_C - t_D$ for the ball to reach Tom is $\frac{x_0}{\frac{3}{5}c} = \frac{5x_0}{3c}$. The difference between these two time intervals $t_C - t_S$ is our given T.

$$T = t_C - t_S = (t_C - t_D) - (t_S - t_D) = \frac{5x_0}{3c} - \frac{x_0}{c} = \frac{2}{3}\frac{x_0}{c} \tag{1}$$

$$\boxed{x_0 = \frac{3}{2}cT} \tag{2}$$

Now that we know x_0, we can use it to solve for t_0, using $t_S - t_D = 0 - t_0 = \frac{x_0}{c}$.

$$\boxed{t_0 = -\frac{3}{2}T} \tag{3}$$

Tom sees Dick throw the ball at $t = 0$, but the time Dick actually throws the ball is before - thus the negative t_0.

b. Where, x_0', and when, t_0', was the ball thrown in Harry's frame? (If you cannot get part a, you can express your answer in terms of x_0 and/or t_0 for partial credit.)

We transform the event (x_0, t_0) into Harry's frame S' with a Lorentz boost $\beta = \frac{4}{5}$, and $\gamma = \frac{5}{3}$.

$$\begin{pmatrix} ct_0' \\ x_0' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct_0 \\ x_0 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} & -\frac{4}{3} \\ -\frac{4}{3} & \frac{5}{3} \end{pmatrix} \begin{pmatrix} ct_0 \\ x_0 \end{pmatrix} \tag{4}$$

$$ct_0' = \frac{5}{3}ct_0 - \frac{4}{3}x_0, \quad x_0' = \frac{5}{3}x_0 - \frac{4}{3}ct_0 \tag{5}$$

Plugging in our results from part a, $t_0 = -\frac{3}{2}T$ and $x_0 = \frac{3}{2}cT$:

$$\boxed{t_0' = -\frac{9}{2}T}, \quad \boxed{x_0' = \frac{9}{2}cT} \tag{6}$$

c. How long did it take the ball to travel from Dick to Tom in Harry's frame? (If you cannot get part a, you can express your answer in terms of x_0 and/or t_0 for partial credit.)

The two events we consider are Dick throwing and Tom catching the ball. In frame S these have spacetime coordinates $(t_D = t_0, x_D = x_0)$ and $(t_C = T, x_C = 0)$, so the intervals we transform are

$$\Delta t_{C-D} = T - t_0, \quad \Delta x_{C-D} = 0 - x_0 = -x_0 \tag{7}$$

Again using the Lorentz transform $\beta = \frac{4}{5}$, and $\gamma = \frac{5}{3}$,

$$\begin{pmatrix} \Delta ct'_{C-D} \\ \Delta x'_{C-D} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \Delta ct_{C-D} \\ \Delta x_{C-D} \end{pmatrix} = \begin{pmatrix} \frac{5}{3} & -\frac{4}{3} \\ -\frac{4}{3} & \frac{5}{3} \end{pmatrix} \begin{pmatrix} c(T - t_0) \\ -x_0 \end{pmatrix} \tag{8}$$

$$\Delta t'_{C-D} = \frac{5}{3}(T - t_0) + \frac{4}{3}\frac{x_0}{c} \tag{9}$$

and again plugging in $t_0 = -\frac{3}{2}T$ and $x_0 = \frac{3}{2}cT$:

$$\Delta t'_{C-D} = \frac{5}{3} \cdot (1 - (-\frac{3}{2}))T + \frac{4}{3} \cdot \frac{3}{2}T = (\frac{5}{3} \cdot \frac{5}{2} + \frac{4}{3} \cdot \frac{3}{2})T \tag{10}$$

gives us the result we're looking for:

$$\boxed{\Delta t'_{C-D} = \frac{37}{6}T}. \tag{11}$$

Problem 3

a) In frame S, the fragment of mass has speed u and it is moving off at an angle θ with respect to the x axis. Therefore, it has velocity $\vec{u} = u_x\hat{x} + u_y\hat{y} = u\cos\theta\hat{x} + u\sin\theta\hat{y}$, where the speed is $u = \sqrt{\vec{u}^{\,2}} = \sqrt{\vec{u}\cdot\vec{u}}$, and the angle θ can be written as $\tan\theta = \frac{u_y}{u_x}$. Frame S' is moving in the $+x$ direction with speed $v = \beta c$ with respect to frame S. We can solve this problem using one of several approaches.

Approach 1:
We can write the velocity of the fragment in frame S' in terms of the velocity of the fragment in frame S using the velocity transformation

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}} = \frac{u\cos\theta - v}{1 - \frac{vu\cos\theta}{c^2}} \tag{1}$$

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} = \frac{u\sin\theta}{\gamma\left(1 - \frac{vu\cos\theta}{c^2}\right)} \tag{2}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. In frame S', the fragment is moving off at an angle θ' with respect to the x' axis, so

$$\tan\theta' = \frac{u'_y}{u'_x} = \frac{u_y}{\gamma(u_x - v)} \tag{3}$$

$$= \frac{u\sin\theta}{\gamma(u\cos\theta - \beta c)} \tag{4}$$

Approach 2:
The velocity of the fragment is a constant. Therefore, in frame S, the components of the velocity of the fragment can be written as $u_x = \frac{\Delta x}{\Delta t}$ and $u_y = \frac{\Delta y}{\Delta t}$, where Δx and Δy are the change in the spatial coordinates of the fragment over a time Δt. The angle θ that the fragment makes with the x axis is given by $\tan\theta = \frac{u_y}{u_x} = \frac{\Delta y}{\Delta x}$. In frame S', the components of the velocity of the fragment can be written as $u'_x = \frac{\Delta x'}{\Delta t'}$ and $u'_y = \frac{\Delta y'}{\Delta t'}$, where $\Delta x'$ and $\Delta y'$ are the change in the spatial coordinates of the fragment over a time $\Delta t'$. The angle θ' that the fragment makes with the x' axis is given by $\tan\theta' = \frac{u'_y}{u'_x} = \frac{\Delta y'}{\Delta x'}$. We can write the change in the fragment's spacetime coordinates in frame S' in terms of those in frame S using the Lorentz transformation

$$c\Delta t' = \gamma(c\Delta t - \beta\Delta x) \tag{5}$$

$$\Delta x' = \gamma(\Delta x - \beta c\Delta t) \tag{6}$$

$$\Delta y' = \Delta y \tag{7}$$

where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. In frame S', $\tan\theta'$ is given by

$$\begin{aligned}
\tan\theta' &= \frac{u'_y}{u'_x} = \frac{\Delta y'}{\Delta x'} && (8)\\
&= \frac{\Delta y}{\gamma(\Delta x - \beta c \Delta t)} = \frac{\frac{\Delta y}{\Delta t}}{\gamma\left(\frac{\Delta x}{\Delta t} - \beta c\right)} = \frac{u_y}{\gamma(u_x - \beta c)} && (9)\\
&= \frac{u\sin\theta}{\gamma(u\cos\theta - \beta c)} && (10)
\end{aligned}$$

Approach 3:
In frame S, the fragment of mass has four-momentum

$$\begin{aligned}
\mathbf{p} &= \begin{pmatrix} \frac{E}{c} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \gamma(\vec{u})mc \\ \gamma(\vec{u})m\vec{u} \end{pmatrix} && (11)\\
&= \begin{pmatrix} \frac{E}{c} \\ p_x \\ p_y \end{pmatrix} = \begin{pmatrix} \gamma(\vec{u})mc \\ \gamma(\vec{u})mu_x \\ \gamma(\vec{u})mu_y \end{pmatrix} && (12)
\end{aligned}$$

where $\gamma(\vec{u}) = \left(1 - \frac{\vec{u}^2}{c^2}\right)^{-1/2}$. In frame S, the fragment is moving off at an angle θ with respect to the x axis, so $\tan\theta = \frac{u_y}{u_x} = \frac{p_y}{p_x}$.
The four-momentum of the fragment in frame S' is

$$\begin{aligned}
\mathbf{p}' &= \begin{pmatrix} \frac{E'}{c} \\ \vec{p}\,' \end{pmatrix} = \begin{pmatrix} \gamma(\vec{u}\,')mc \\ \gamma(\vec{u}\,')m\vec{u}\,' \end{pmatrix} && (13)\\
&= \begin{pmatrix} \frac{E'}{c} \\ p'_x \\ p'_y \end{pmatrix} = \begin{pmatrix} \gamma(\vec{u}\,')mc \\ \gamma(\vec{u}\,')mu'_x \\ \gamma(\vec{u}\,')mu'_y \end{pmatrix} && (14)
\end{aligned}$$

where $\gamma(\vec{u}\,') = \left(1 - \frac{\vec{u}\,'^2}{c^2}\right)^{-1/2}$.

We can write the four-momentum of the fragment in frame S' in terms of its four-momentum in frame S using the Lorentz transformation

$$\begin{aligned}
\mathbf{p}' &= \Lambda(\beta\hat{x})\mathbf{p} && (15)\\
\begin{pmatrix} \frac{E'}{c} \\ p'_x \\ p'_y \end{pmatrix} &= \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{E}{c} \\ p_x \\ p_y \end{pmatrix} && (16)\\
&= \begin{pmatrix} \gamma\left(\frac{E}{c} - \beta p_x\right) \\ \gamma\left(p_x - \beta\frac{E}{c}\right) \\ p_y \end{pmatrix} && (17)
\end{aligned}$$

where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

In frame S', the fragment is moving off at an angle θ' with respect to the x' axis, so

$$\begin{aligned}
\tan\theta' &= \frac{u'_y}{u'_x} = \frac{p'_y}{p'_x} && (18)\\
&= \frac{p_y}{\gamma\left(p_x - \beta\frac{E}{c}\right)} = \frac{p_y/p_x}{\gamma\left(1 - \beta\frac{E/c}{p_x}\right)} && (19)\\
&= \frac{\tan\theta}{\gamma\left(1 - \beta\frac{c}{u_x}\right)} = \frac{\tan\theta}{\gamma\left(1 - \beta\frac{c}{u\cos\theta}\right)} = \frac{u\cos\theta\tan\theta}{\gamma(u\cos\theta - \beta c)} && (20)\\
&= \frac{u\sin\theta}{\gamma(u\cos\theta - \beta c)} && (21)
\end{aligned}$$

b) The problem statement has an error: the expression given is actually the minimum value of $\tan\theta'$, not the maximum value. The expression for the maximum value of $\tan\theta'$ is identical except that there is no negative sign in front.

The fragment velocity $u < v$ and the angle θ is in the range $0 \le \theta \le 2\pi$. Therefore, physically, θ' must be in the range $\frac{\pi}{2} < \theta'_{\text{min}} \le \theta' \le \theta'_{\text{max}} < \frac{3\pi}{2}$. In this region, $\tan\theta'$ is a monotonically increasing function of θ', and $(\tan\theta')_{\text{min}} \le \tan\theta' \le (\tan\theta')_{\text{max}}$. The angle $\theta' = 0$ when $\theta = 0$, π, or 2π. As θ goes from 0 to 2π, the angle θ' starts at π, then decreases to a minimum of θ'_{min}, then increases back to π, then increases to a maximum of θ'_{max}, then decreases back to π. Similarly, as θ goes from 0 to 2π, $\tan\theta'$ starts at 0, then decreases to a minimum of $(\tan\theta')_{\text{min}}$, then increases back to 0, then increases to a maximum of $(\tan\theta')_{\text{max}}$, then decreases back to 0.

We can find the extrema of $\tan\theta'$ by differentiating it with respect to θ and setting the result equal to zero. The derivative of $\tan\theta'$ with respect to θ is

$$\begin{aligned}
\frac{d}{d\theta}\tan\theta' &= \frac{d}{d\theta}\frac{u\sin\theta}{\gamma(u\cos\theta - \beta c)} && (22)\\
&= u\cos\theta\frac{1}{\gamma(u\cos\theta - \beta c)} + u\sin\theta(-1)\frac{-u\sin\theta}{\gamma(u\cos\theta - \beta c)^2} && (23)\\
&= \frac{u\cos\theta}{\gamma(u\cos\theta - \beta c)} + \frac{u^2\sin^2\theta}{\gamma(u\cos\theta - \beta c)^2} && (24)\\
&= \frac{u\cos\theta(u\cos\theta - \beta c) + u^2\sin^2\theta}{\gamma(u\cos\theta - \beta c)^2} && (25)\\
&= \frac{u^2 - \beta c u\cos\theta}{\gamma(u\cos\theta - \beta c)^2} && (26)
\end{aligned}$$

Since $u < v = \beta c$, the denominator is never equal to zero. Let $\tan\theta'$ have an extremum at $\theta = \theta^*$. Then,

$$\begin{aligned}
0 &= \left.\frac{d}{d\theta}\tan\theta'\right|_{\theta=\theta^*} = \frac{u^2 - \beta c u\cos\theta^*}{\gamma(u\cos\theta^* - \beta c)^2} && (27)\\
0 &= u^2 - \beta c u\cos\theta^* && (28)\\
\cos\theta^* &= \frac{u}{\beta c} = \frac{u}{v} && (29)
\end{aligned}$$

This expression is bounded by $0 < \cos\theta^* = \frac{u}{v} < 1$. Since $\cos\theta$ is positive in the ranges $0 < \theta < \frac{\pi}{2}$ and $\frac{3\pi}{2} < \theta < 2\pi$, and both ranges are physically possible, there are two angles θ_1^* and θ_2^* at which $\cos\theta_1^* = \cos\theta_2^* = \frac{u}{v}$, and thus at which $\tan\theta'$ is an extremum. The two angles are related by $\theta_2^* = 2\pi - \theta_1^*$. Let angle θ_1^* be in the range $0 < \theta_1^* < \frac{\pi}{2}$. The sin of this angle is $\sin\theta_1^* = \frac{\sqrt{v^2-u^2}}{v} = \sqrt{1-\frac{u^2}{v^2}}$. Then, angle θ_2^* is in the range $\frac{3\pi}{2} < \theta_2^* < 2\pi$. The sin of this angle is $\sin\theta_2^* = -\sin\theta_1^* = -\sqrt{1-\frac{u^2}{v^2}}$. From our earlier discussion, we see that θ_1^* gives $\tan\theta'$ its minimum value, and θ_2^* gives $\tan\theta'$ its maximum value, though we can show this explicitly.

The minimum value of $\tan\theta'$ is

$$
\begin{aligned}
\tan\theta'_{\text{min}} &= \frac{u\sin\theta_1^*}{\gamma(u\cos\theta_1^* - \beta c)} && (30)\\
&= \frac{1}{\gamma}\frac{u\sqrt{1-\frac{u^2}{v^2}}}{u\frac{u}{v} - v} && (31)\\
&= \frac{1}{\gamma}\frac{u}{v}(-1)\frac{\sqrt{1-\frac{u^2}{v^2}}}{1-\frac{u^2}{v^2}} && (32)\\
&= -\frac{1}{\gamma}\frac{u}{v}\left(1-\frac{u^2}{v^2}\right)^{-1/2} && (33)\\
&= -\left(1-\frac{v^2}{c^2}\right)^{1/2}\left(1-\frac{u^2}{v^2}\right)^{-1/2}\frac{u}{v} && (34)
\end{aligned}
$$

Since $\tan\theta'_{\text{min}} < 0$, the angle θ'_{min} is in the range $\frac{\pi}{2} < \theta'_{\text{min}} < \pi$.

Similarly, the maximum value of $\tan\theta'$ is

$$
\begin{aligned}
\tan\theta'_{\text{max}} &= \frac{u\sin\theta_2^*}{\gamma(u\cos\theta_2^* - \beta c)} && (35)\\
&= -\frac{u\sin\theta_1^*}{\gamma(u\cos\theta_1^* - \beta c)} && (36)\\
&= -(\tan\theta')_{\text{min}} && (37)\\
&= \left(1-\frac{v^2}{c^2}\right)^{1/2}\left(1-\frac{u^2}{v^2}\right)^{-1/2}\frac{u}{v} && (38)
\end{aligned}
$$

Since $\tan\theta'_{\text{max}} > 0$, the angle θ'_{max} is in the range $\pi < \theta'_{\text{max}} < \frac{3\pi}{2}$.

Problem 4

By conservation of 4-momentum:

$$\begin{pmatrix} E_M \\ \vec{p}_M c \end{pmatrix} = \begin{pmatrix} E_1 \\ \vec{p}_1 c \end{pmatrix} + \begin{pmatrix} E_2 \\ \vec{p}_2 c \end{pmatrix}$$

Eliminating the second particle by using the invariant scalar product:

$$\begin{aligned} m^2c^4 &= (E_2)^2 - (\vec{p}_2 c)\cdot(\vec{p}_2 c) \\ &= (E_M - E_1)^2 - (\vec{p}_M c - \vec{p}_1 c)\cdot(\vec{p}_M c - \vec{p}_1 c) \\ &= E_M^2 - 2E_M E_1 + E_1^2 - p_M^2 c^2 + 2\vec{p}_M \cdot \vec{p}_1 c^2 - p_1^2 c^2 \\ &= M^2c^4 + m^2c^4 - 2E_M E_1 + 2p_M p_1 c^2 \cos\theta \end{aligned}$$

Isolating the energy of the first particle:

$$\begin{aligned} 2E_M E_1 &= M^2c^4 + 2p_M p_1 c^2 \cos\theta \\ 4E_M^2 E_1^2 &= M^4c^8 + 4M^2c^6 p_M p_1 \cos\theta + 4p_M^2 p_1^2 c^4 \cos^2\theta \end{aligned}$$

Rewriting in terms of momentum:

$$4E_M^2 p_1^2 c^2 + 4E_M^2 m^2 c^4 = M^4c^8 + 4M^2c^6 p_M p_1 \cos\theta + 4p_M^2 p_1^2 c^4 \cos^2\theta$$

$$4\left(E_M^2 - p_M^2 c^2 \cos^2\theta\right) p_1^2 - 4M^2c^4 p_M \cos\theta \, p_1 - \left(M^4c^6 - 4E_M^2 m^2 c^2\right) = 0$$

By the quadratic formula:

$$p_1 = \frac{M^2c^4 p_M \cos\theta \pm \sqrt{M^4c^8 p_M^2 \cos^2\theta + \left(E_M^2 - p_M^2c^2\cos^2\theta\right)\left(M^4c^6 - 4E_M^2 m^2 c^2\right)}}{2\left(E_M^2 - p_M^2 c^2 \cos^2\theta\right)}$$

Only the + root actually gives a positive p_1 (a negative p_1 means the particle is actually traveling the opposite direction).

Note that the algebra is much simpler if it is assumed that $E_1 \approx p_1 c$; i.e. the ultra-relativistic limit:

$$2E_M p_1 c - 2p_M p_1 c^2 \cos\theta \approx M^2c^4$$

$$p_1 \approx \frac{Mc^2}{2\left(E_M - p_M c \cos\theta\right)} Mc$$

It can also be verified that the above reduces to this in the $m \ll M/2$ limit:

$$\begin{aligned} p_1 &\approx \frac{M^2c^4 p_M \cos\theta + \sqrt{M^4c^8 p_M^2 \cos^2\theta + \left(E_M^2 - p_M^2c^2\cos^2\theta\right)\left(M^4c^6\right)}}{2\left(E_M^2 - p_M^2 c^2 \cos^2\theta\right)} \\ &= \frac{M^2c^4 p_M \cos\theta + \sqrt{E_M^2 M^4 c^6}}{2\left(E_M^2 - p_M^2 c^2 \cos^2\theta\right)} \\ &= \frac{\left(Mc^2\right)(Mc)\left(p_M c\cos\theta + E_M\right)}{2\left(E_M^2 - p_M^2 c^2 \cos^2\theta\right)} \\ &= \frac{Mc^2}{2\left(E_M - p_M c \cos\theta\right)} Mc \end{aligned}$$

In all cases, the substitution $E_M = \sqrt{p_M^2 c^2 + M^2 c^4}$ needs to be made to get the final answer in terms of the right variables.

Problem 5:

According to Babinet's principle (and remember it is a vector subtraction):

$$\overrightarrow{E_C}(\vec{r},t) = E_0\sin(kx-\omega t)\,\hat{y} - E_0\frac{\sin\left(\frac{\beta}{2}\right)}{\frac{\beta}{2}}\sin(kr-\omega t)\widehat{r_\perp}$$

Where $\widehat{r_\perp}$ is the unit vector pointing to direction perpendicular to $\hat{r}$. This is the polarization of $\overrightarrow{E_s}$. $\hat{y}$ is the polarization of $\overrightarrow{E_\iota}$.

At the screen we have:

$$x = L, r = \frac{L}{\cos\theta}, \beta = kDsin\theta$$

$$\overrightarrow{E_C}(\theta,t)_{screen} = E_0\sin(kL-\omega t)\hat{y} - E_0\frac{\sin\left(\frac{\beta}{2}\right)}{\frac{\beta}{2}}\sin\left(k\frac{L}{\cos\theta}-\omega t\right)\widehat{r_\perp}$$

Subtract by component:

$$\left[\overrightarrow{E_C}(\theta,t)_{screen}\right]_y = E_0\sin(kL-\omega t) - E_0\frac{\sin\left(\frac{\beta}{2}\right)}{\frac{\beta}{2}}\sin\left(k\frac{L}{\cos\theta}-\omega t\right)\cos\theta$$

$$\left[\overrightarrow{E_C}(\theta,t)_{screen}\right]_x = -E_0\frac{\sin\left(\frac{\beta}{2}\right)}{\frac{\beta}{2}}\sin\left(k\frac{L}{\cos\theta}-\omega t\right)\sin\theta$$

Therefore:

$$\overrightarrow{E_C}(\theta,t)_{screen}^2 = \left[\overrightarrow{E_C}(\theta,t)_{screen}\right]_x^2 + \left[\overrightarrow{E_C}(\theta,t)_{screen}\right]_y^2$$

$$= E_0^2\left[\frac{\sin\left(\frac{\beta}{2}\right)}{\frac{\beta}{2}}\right]^2\sin^2\left(k\frac{L}{\cos\theta}-\omega t\right)\sin^2\theta + E_0^2\sin^2(kL-\omega t)$$

$$+ E_0^2\left[\frac{\sin\left(\frac{\beta}{2}\right)}{\frac{\beta}{2}}\right]^2\sin^2\left(k\frac{L}{\cos\theta}-\omega t\right)\cos^2\theta$$

$$-2E_0^2\frac{\sin\left(\frac{\beta}{2}\right)}{\frac{\beta}{2}}\cos\theta\sin\left(k\frac{L}{\cos\theta}-\omega t\right)\sin(kD-\omega t)$$

Note the identity for the last term:

$$\sin\left(k\frac{L}{\cos\theta}-\omega t\right)\sin(kL-\omega t)$$

$$=\frac{1}{2}\left[\cos kL\left(\frac{1}{\cos\theta}-1\right)-\cos\left(\frac{kL}{\cos\theta}+kL-2\omega t\right)\right]$$

Applying time average for each term, we have:

$$\langle\overrightarrow{E_C}(\theta,t)^2_{screen}\rangle=\frac{1}{2}E_0^2\left[\frac{\sin\left(\frac{\beta}{2}\right)}{\frac{\beta}{2}}\right]^2\sin^2\theta+\frac{1}{2}E_0^2+\frac{1}{2}E_0^2\left[\frac{\sin\left(\frac{\beta}{2}\right)}{\frac{\beta}{2}}\right]^2\cos^2\theta$$
$$-E_0^2\frac{\sin\left(\frac{\beta}{2}\right)}{\frac{\beta}{2}}\cos\theta\cos kL\left(\frac{1}{\cos\theta}-1\right)$$

To simplify:

$$\langle\overrightarrow{E_C}(\theta,t)^2_{screen}\rangle=\frac{1}{2}E_0^2\left\{\left[\frac{\sin\left(\frac{\beta}{2}\right)}{\frac{\beta}{2}}\right]^2+1-\frac{2\sin\left(\frac{\beta}{2}\right)}{\frac{\beta}{2}}\cos\theta\cos kL\left(\frac{1}{\cos\theta}-1\right)\right\}$$

Therefore, from definition of intensity, we have:

$$I(\theta)=\langle S_C(\theta,t)_{screen}\rangle=\epsilon_0 c\langle\overrightarrow{E_C}(\theta,t)^2_{screen}\rangle$$
$$=\frac{\epsilon_0 c}{2}E_0^2\left\{\left[\frac{\sin\left(\frac{\beta}{2}\right)}{\frac{\beta}{2}}\right]^2+1-\frac{2\sin\left(\frac{\beta}{2}\right)}{\frac{\beta}{2}}\cos\theta\cos kL\left(\frac{1}{\cos\theta}-1\right)\right\}$$