CBE 150A - Spring 2016 Midterm 2
Problem 1

$V_x(y)$

a) $0 = -\frac{\partial P}{\partial x} + \eta \frac{d^2 V_x}{dy^2}$ $(\Delta P \equiv P_1 - P_0)$

$\frac{d^2 V_x}{dy^2} = \frac{(P_0 - P_1)}{\eta L}$ $\int \Rightarrow$ $\frac{dV_x}{dy} = -\frac{\Delta P}{\eta L} y + C_1$

$\int \Rightarrow$ $V_x = -\frac{\Delta P}{2\eta L} y^2 + C_1 y + C_2$

$V_x\Big|_{y=0} = C_2 = U$

$V_x\Big|_{y=h} = -\frac{\Delta P}{2\eta L} h^2 + C_1 h + U = 0$

$C_1 = \frac{\Delta P h}{2\eta L} - \frac{U}{h}$

$V_x = -\frac{\Delta P}{2\eta L} y^2 + \left(\frac{\Delta P h}{2\eta L} - \frac{U}{h}\right) y + U$

Forward $\boxed{V_x = U\left(1 - \frac{y}{h}\right) + \frac{\Delta P h^2}{2\eta L_1}\left[\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2\right]}$

$\Delta P = P_1 - P_2$

b) Back: $V_x = U\left(1 - \frac{y}{h}\right) - \frac{\Delta P h^2}{2\eta L_2}\left[\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2\right]$

To determine L_2: $\int V_x \, dy = 0$
(Backward)

Note: same velocity profile with negative pressure drop

$$\int_0^h v_x dy = U\left[y - \frac{y^2}{2h}\right] - \frac{\Delta P h^2}{2\eta L_2}\left[\frac{y^2}{2h} - \frac{y^3}{3h^2}\right]\Bigg|_0^h$$

$$= U\left[h - \frac{h}{2}\right] - \frac{\Delta P h^2}{2\eta L_2}\left[\frac{h}{2} - \frac{h}{3}\right]$$

$$= \frac{Uh}{2} - \frac{\Delta P h^3}{12\eta L_2} = 0$$

$$L_2 = \frac{2}{Uh} \cdot \frac{\Delta P h^3}{12\eta} = \boxed{\frac{\Delta P h^2}{6\eta U}}$$

$\Delta P = P_1 - P_2$

c) $Uh_\infty = \int_0^y v_x\, dy$ (forward flow)

$$\int v_x dy = \frac{Uh}{2} + \frac{\Delta P h^3}{12\eta L_1} = Uh_\infty$$

$$\boxed{h_\infty = \frac{h}{2} + \frac{\Delta P h^3}{12\eta L_1 U}}$$

d). $F = \tau_{yx}^b \cdot L_2 + \tau_{yx}^f L_1$

$$= \eta \frac{dV_x^b}{dy}\Bigg|_{y=0} \cdot L_2 + \eta \frac{dV_x^f}{dy}\Bigg| L_1$$

$$= \eta\left[-\frac{U}{h} - \frac{\Delta P h}{2\eta L_2}\right] L_2 + \eta\left[-\frac{U}{h} + \frac{\Delta P h}{2\eta L_1}\right] L_1$$

$$F = -\frac{\eta U}{h}(L_1 + L_2)$$

(−x direction)

NAME Solutions

CHEMICAL ENGINEERING 150A
Mid-term Examination 2: PART B

Write down all assumptions made and show your work.
The exam is closed book and closed note except for your 2 pages of notes (your 1 page from midterm 1, plus 1 page from new material).

There are 3 problems (1 in part A, 2 in part B). Please note the point values for each problem in the table below and plan your time accordingly. TRY ALL PARTS. Even on multi-part problems, later sections may be independent of initial sections.

Part B of your exam should have **9 numbered pages including this coversheet.**

TABLE 7-1
CONTINUITY EQUATION IN RECTANGULAR, CYLINDRICAL, AND SPHERICAL COORDINATES

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})$$

Rectangular (x, y, z) coordinates:

$$\nabla \cdot (\rho \mathbf{v}) = \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)$$

Cylindrical (r, θ, z) coordinates:

$$\nabla \cdot (\rho \mathbf{v}) = \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z)$$

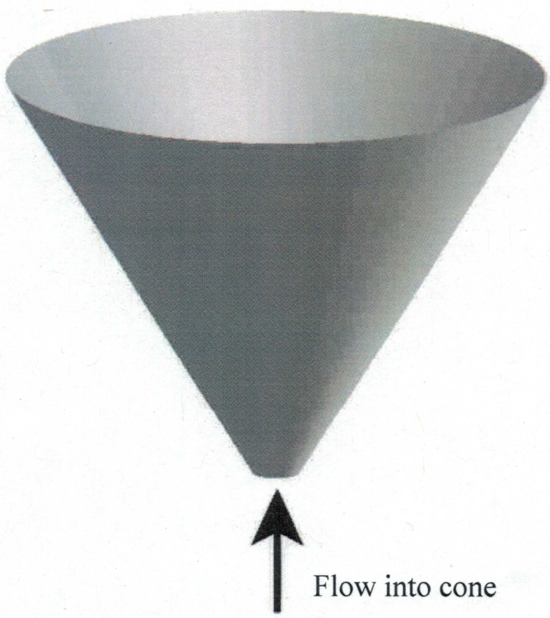
Spherical (r, θ, ϕ) coordinates:

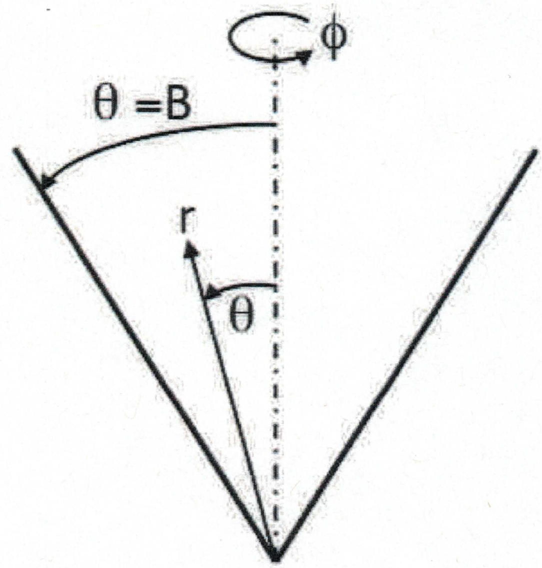
$$\nabla \cdot (\rho \mathbf{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}(\rho v_\phi)$$

NAME <u>Solutions</u>

Problem 2

An incompressible, Newtonian fluid flows into a cone through a narrow opening at the vertex as depicted below. We wish to solve for the velocity profile within the cone. Cones may be represented in spherical coordinates by a single value of θ, i.e. $\theta = B$ (a constant), rotated about the ϕ-axis (out of the page). Neglect entrance effects (near r=0).





a) Write down your hypothesis for the velocity components present in this problem, and what they are a function of.

$V_r = V_r(r, \theta)$

$V_\theta = 0$

$V_\phi = 0$

b) Simplify the equation of continuity for your assumption. What can you determine from the continuity equation? (See page 1 for the continuity equations).

$$0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right)$$

$$\boxed{V_r = \frac{f(\theta)}{r^2}}$$

NAME Solutions

c) Considering the CREEPING FLOW LIMIT, cross out the terms in the Navier Stokes Equations that are not necessary given your assumptions in part a.

TABLE 7-10
NAVIER–STOKES EQUATIONS FOR A NEWTONIAN FLUID WITH A CONSTANT VISCOSITY IN SPHERICAL (r, θ, ϕ) COORDINATES[a]

r component: $\cancel{\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)}$

$$= -\frac{\partial \mathcal{P}}{\partial r} + \eta\Big[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right) + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}}$$
$$- \frac{2}{r^2}v_r - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}} - \cancel{\frac{2}{r^2}v_\theta\cot\theta} - \cancel{\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}}\Big]$$

θ component: $\cancel{\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right)}$

$$= -\frac{1}{r}\frac{\partial \mathcal{P}}{\partial \theta} + \eta\Big[\cancel{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)} + \cancel{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_\theta}{\partial \theta}\right)} + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2}}$$
$$+ \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \cancel{\frac{v_\theta}{r^2\sin^2\theta}} - \cancel{\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}}\Big]$$

ϕ component: $\cancel{\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r}\cot\theta\right)}$

$$= -\cancel{\frac{1}{r\sin\theta}\frac{\partial \mathcal{P}}{\partial \phi}} + \eta\Big[\cancel{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right)} + \cancel{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_\phi}{\partial \theta}\right)} + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2}}$$
$$- \cancel{\frac{v_\phi}{r^2\sin^2\theta}} + \cancel{\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi}} + \cancel{\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}}\Big]$$

[a]The equations are written in terms of the equivalent pressure, $\mathcal{P}$.

d) Using the continuity equation, simplify the components of the Navier-Stokes equation and eliminate the pressure term(s) to obtain an **ordinary differential equation**. Circle the equation. YOU DO NOT NEED TO SOLVE THE EQUATION.

θ-comp: $0 = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \eta\frac{2}{r^2}\frac{\partial V_r}{\partial \theta}$, $\quad V_r = \frac{f(\theta)}{r^2}$

$\frac{\partial P}{\partial \theta} = \frac{2}{r^3}\eta\frac{df}{d\theta} \longrightarrow \underline{\frac{\partial P}{\partial \theta} = \frac{2\eta}{r^3}f'}$

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r-comp: $0 = -\frac{\partial P}{\partial r} + \eta\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v_r}{\partial \theta}\right) - \frac{2}{r^2} v_r\right]$

$$\frac{\partial P}{\partial r} = \eta\left[\cancel{\frac{f}{r^2}\frac{\partial}{\partial r}\left(r^3 \cdot \frac{-2}{r^3}\right)} + \frac{f'}{r^4 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta) + \frac{f''}{r^4} - \cancel{\frac{2f}{r^4}}\right]$$

$$\frac{\partial P}{\partial r} = \eta\left[\frac{f'}{r^4 \sin\theta}\cos\theta + f''/r^4\right]$$

$$\underline{\frac{\partial P}{\partial r} = \frac{\eta}{r^4}\left[f'' + \cot\theta \cdot f'\right]}$$

Cross differentiate:

$$\frac{\partial}{\partial r}\left(\frac{\partial P}{\partial \theta}\right) = 2f'\eta \frac{\partial}{\partial r}(r^{-3}) = -\frac{6\eta}{r^4} f'$$

$$\frac{\partial}{\partial \theta}\left(\frac{\partial P}{\partial r}\right) = \frac{\eta}{r^4}\left[f''' + f''\cot\theta - f'\csc^2\theta\right]$$

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equate: $-\frac{6\cancel{\eta}}{\cancel{r^4}} f' = \frac{\cancel{\eta}}{\cancel{r^4}}\left[f''' + f''\cot\theta - f'\csc^2\theta\right]$

$$\boxed{0 = f''' + f''\cot\theta + f'(6 - \csc^2\theta)}$$

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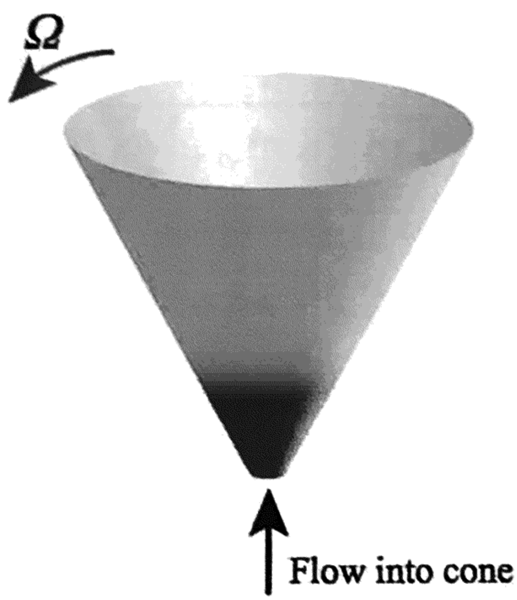
e) Now consider that the cone is rotating as shown below. The cone rotates at angular speed Ω, and the velocity of the cone at some r is given by $V = \Omega r \sin B$.



Write down your hypothesis for the velocity components present in this problem, and what they are a function of.

$$v_r = v_r(r, \theta)$$

$$v_\phi = v_\phi(r, \theta)$$

$$v_\theta = 0$$

f) Simplify the equation of continuity for your assumption. What can you determine from the continuity equation?

$$0 = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_r)$$

$$r^2 v_r \neq f(r)$$

or

$$v_r = \frac{f(\theta)}{r^2}$$

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g) Considering the most general case (i.e., do NOT assume creeping flow), cross out the terms in the Navier Stokes Equations that are not necessary based on your assumptions in e.

TABLE 7-10
NAVIER–STOKES EQUATIONS FOR A NEWTONIAN FLUID WITH A CONSTANT VISCOSITY IN SPHERICAL (r, θ, ϕ) COORDINATES[a]

r component:
$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + v_r\frac{\partial v_r}{\partial r} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} + \cancel{\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}} - \frac{\cancel{v_\theta^2} + v_\phi^2}{r}\right)$$
$$= -\frac{\partial \mathcal{P}}{\partial r} + \eta\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}}\right.$$
$$\left. - \frac{2}{r^2}v_r - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}} - \cancel{\frac{2}{r^2}v_\theta\cot\theta} - \cancel{\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}}\right]$$

θ component:
$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r\frac{\partial v_\theta}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}} + \cancel{\frac{v_r v_\theta}{r}} - \frac{v_\phi^2\cot\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial \mathcal{P}}{\partial\theta} + \eta\left[\cancel{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)} + \cancel{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_\theta}{\partial\theta}\right)} + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2}}\right.$$
$$\left. + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \cancel{\frac{v_\theta}{r^2\sin^2\theta}} - \cancel{\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial\phi}}\right]$$

ϕ component:
$$\rho\left(\cancel{\frac{\partial v_\phi}{\partial t}} + v_r\frac{\partial v_\phi}{\partial r} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}} + \cancel{\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}} + \frac{v_\phi v_r}{r} + \cancel{\frac{v_\theta v_\phi}{r}\cot\theta}\right)$$
$$= -\cancel{\frac{1}{r\sin\theta}\frac{\partial \mathcal{P}}{\partial\phi}} + \eta\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_\phi}{\partial\theta}\right) + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2}}\right.$$
$$\left. - \frac{v_\phi}{r^2\sin^2\theta} + \cancel{\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi}} + \cancel{\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\phi}}\right]$$

[a]The equations are written in terms of the equivalent pressure, $\mathcal{P}$.

h) What is the total number of boundary conditions needed to solve this problem (including those for pressure)? Write the boundary conditions in the theta direction for each of the velocity components.

10 total

$\theta = 0$ $\quad \frac{\partial v_r}{\partial \theta} = 0$, $\quad v_\phi = 0$ or finite

$\theta = \beta$ $\quad v_r = 0$ $\quad v_\phi = \Omega r \sin\beta$

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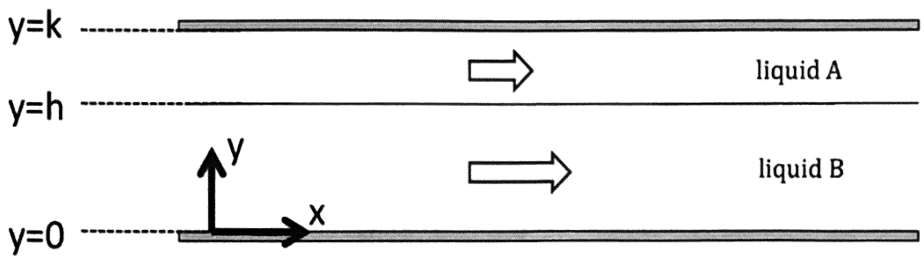
Problem 3

Briefly answer the following questions

a) Two immiscible, incompressible, Newtonian liquids A and B are flowing in laminar flow between two parallel plates, as shown below, under the influence of a pressure gradient.



The velocity field in liquid A and B can be written as:

$$v_x^A = \frac{1}{2\eta}\frac{\Delta P}{L}y^2 + C_1 y + C_2 \quad \text{and} \quad v_x^B = \frac{1}{2\eta}\frac{\Delta P}{L}y^2 + C_3 y + C_4$$

where C_1, C_2, C_3, and C_4 are constants of integration.

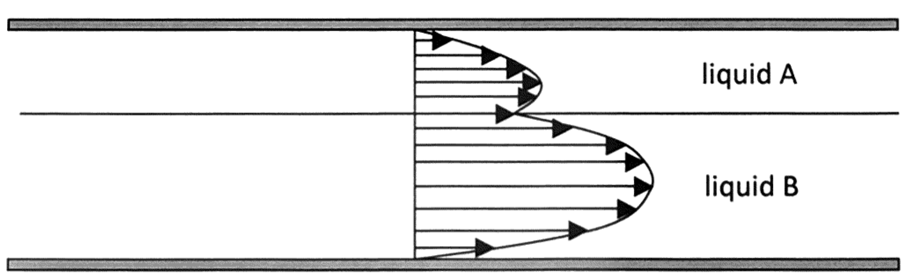
Write down the boundary conditions needed to complete the solution to this problem.

$y = 0 \quad v_x^B = 0$

$y = h \quad v_x^A = v_x^B \ , \quad \tau_{yx}^A = \tau_{yx}^B$

$y = k \quad v_x^A = 0$

b) Would there ever be the possibility that the velocity profiles would be of the form shown in the figure below? Explain briefly the reasons for your answer.



No $\quad \tau_{yx}^A \neq \tau_{yx}^B$

Shear Stress must be continuous at the fluid-fluid interface, but here it is not. The velocity gradient in A is the opposite sign of that in B, and the viscosity cannot be negative.

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c) In a general flow at high Reynolds numbers near a solid surface, which of the following are true (circle all that apply): (NOTE: Points will be subtracted for incorrect answers).

1. viscous effects can be neglected throughout the flow
2. inertial effects can be neglected throughout the flow
3. the pressure gradient terms can be neglected throughout the flow
4. the fluid may be treated as inviscid everywhere except near solid surfaces
5. a boundary layer forms near solid surfaces, and the flow inside the boundary layer may be treated as a potential flow
6. a boundary layer forms near solid surfaces, and the flow outside the boundary layer may be treated as a potential flow
7. inertial effects can be neglected inside the boundary layer
8. the pressure gradient terms can be neglected outside the boundary layer
9. the thickness of the boundary layer increases with increasing viscosity
10. None of the above.

d) In a creeping flow, which of the following are true (circle all that apply): (NOTE: Points will be subtracted for incorrect answers).

1. the inertial terms can be neglected throughout the flow
2. the viscous terms can be neglected throughout the flow
3. terms in the Navier-Stokes equations that are multiplied by viscosity can be neglected
4. terms in the Navier-Stokes equations that are multiplied by density can be neglected
5. the no slip boundary conditions cannot be satisfied
6. the flow is reversible, so that if a circle of dye is placed in a flow and the flow is run forwards and then backwards by the same amount, the circle of dye will return to its original position
7. the pressure gradient terms can be neglected
8. None of the above.

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e) On the plot below, where y is the vertical axis and x is the horizontal axis, plot the following equation:

$$y = 1000 / x^3$$