MATH 54 MIDTERM II, XINYI YUAN
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(7 PAGES)

Problem Number	1	2	3	4	5	6	Total
Score							

YOUR NAME: ______________________

Last Name of Your Section Instructor (please circle): Agrawal, Dalal, Drouot, Gleason, Miller, Yott, Yuan, Zhang

Time of your discussion section: ______________

1. (5 points) Find the rank of each of the following matrices. Explain your results briefly.

(a) The 4×4 matrix

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 5 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Sol: An echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The rank is 3.

(b) The 5×3 matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \\ 13 & 14 & 15 \end{pmatrix}.$$

Sol: By the row operations $(R2)-(R1)$, $(R3)-(R1)$, $(R4)-(R1)$ and $(R5)-(R1)$, get

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \\ 9 & 9 & 9 \\ 12 & 12 & 12 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then Rank $= 2$.

(c) For $a < b < c$, the 3×3 matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}.$$

Sol: Take $(R2) - a \cdot (R1)$, $(R3) - a^2 \cdot (R1)$. Get

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b(b-a) & c(c-a) \end{pmatrix}$$

Take $(R3) - b \cdot (R2)$, get

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-b)(c-a) \end{pmatrix}.$$

The diagonal entries are nonzero by $a<b<c$.

So the rank is 3.

2. (5 points) Let V be the vector space of all polynomials $f(t)$ of degrees less than 5 and satisfying $f(-t) = f(t)$. Let $T : V \to V$ be the linear transformation defined by $T(f(t)) = f(t) - f(0)$.

(a) Find the dimension of V.

(b) Describe the kernel and the range of T.

(Note: We have assumed that V is a vector space and that T is a linear transformation. You are not required to prove these two statements, and you can use them freely.)

Sol: (a). Write $f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$

Then $f(-t) = a_0 - a_1 t + a_2 t^2 - a_3 t^3 + a_4 t^4$.

The relation $f(t) = f(-t)$ gives

$a_0 = a_0$, $a_1 = -a_1$, $a_2 = a_2$, $a_3 = -a_3$, $a_4 = a_4$.

So $a_1 = a_3 = 0$, a_0, a_2, a_4 arbitrary.

A basis of V is $\{1, t, t^2\}$.

$\dim V = 3$.

(b). For $f(t) = a_0 + a_2 t^2 + a_4 t^4$,

$T(f(t)) = f(t) - f(0) = a_2 t^2 + a_4 t^4$

If $T(f(t)) = 0$, then $a_2 t^2 + a_4 t^4 = 0$, $a_2 = a_4 = 0$

So $\ker(T) = \text{Span}\{1\} = \{a_0\}$

$f(t) = a_0$

Since a_2, a_4 can be arbitrary,

$\text{range}(T) = \text{Span}\{t^2, t^4\} = \{a_2 t^2 + a_4 t^4\}$.

3. (5 points) Find all values of a such that the matrix

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & a \end{pmatrix}$$

is diagonalizable.

Sol: A is upper triangular

Then the eigenvalues are $1, 2, a$.

①. If $a \neq 1, 2$, all eigenvalues are distinct

A is diagonalizable

②. If $a = 1$, then $\lambda = 1$ has multiplicity 2.

The eigenspace is the solution set of

$$(A - I)\vec{x} = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\vec{x} = 0$$

This gives $\text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$, which is 1-dimensional

We have $1 < 2$. Not diagonalizable

③. If $a = 2$, then $\lambda = 2$ has multiplicity 2

The eigenspace is the solution set of

$$(A - 2I)\vec{x} = 0 \Rightarrow \begin{pmatrix} -1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\vec{x} = 0$$

This gives $\text{Span}\left\{\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$, which is 2-dim'l

We have $2 = 2$. The root $\lambda = 1$ has multiplicity 1, which is not a problem.

So A is diagonalizable in this case.

Conclusion: A is diagonalizable $\Longleftrightarrow$ $\lambda \neq 1$.

4. (5 points) Let W be the subspace of $\mathbb{R}^4$ given by $x_1+x_2+x_3+x_4=0$. Compute the orthogonal projection $\text{proj}_W(\vec{v})$ of the vector $\vec{v}=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ to the space W.

Sol: By definition,

$$W^{\perp} = \text{Span}\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\}.$$

Then

$$\text{proj}_W(\vec{v})$$

$$= \vec{v} - \text{proj}_{W^{\perp}}(\vec{v})$$

$$= \begin{pmatrix}1\\2\\3\\4\end{pmatrix} - \frac{1+2+3+4}{1+1+1+1}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}-3\\-1\\1\\3\end{pmatrix}.$$

Remark: Another way is to find a basis of W by the usual way of solving equations, change it to an orthogonal basis by Gram-Schmidt, and then compute $\text{proj}_W(\vec{v})$ directly using this orthogonal basis.

5. (5 points) Let A be a 2×2 matrix which has two distinct real eigenvalues. Assume that $A^3 + A^2 - A - I_2 = 0_2$. Find all possible forms of the matrix A^2. (Note: I_2 denotes the 2×2 identity matrix, and 0_2 denotes the 2×2 matrix whose entries are all 0.)

Sol: A has 2 distinct real eigenvalues

$\Rightarrow$ A is diagonalizable

Write $A = PDP^{-1}$, $D = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$.

Then

$$A^3 + A^2 - A - I_2$$

$$= PD^3P^{-1} + PD^2P^{-1} - PDP^{-1} - I$$

$$= P(D^3 + D^2 - D - I)P^{-1}$$

Since P is invertible, have

$$D^3 + D^2 - D - I = 0,$$

which is

$$\begin{pmatrix} f(\lambda_1) & \\ & f(\lambda_2) \end{pmatrix} = 0$$

where $f(t) = t^3 + t^2 - t - 1 = (t+1)^2(t-1)$.

Then $f(\lambda_1) = f(\lambda_2) = 0$, which gives $\lambda_1, \lambda_2 = \pm 1$.

Hence, $D^2 = \begin{pmatrix} \lambda_1^2 & \\ & \lambda_2^2 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = I_2$,

$$A^2 = PD^2P^{-1} = PIP^{-1} = I.$$

6. (5 points) Let A and B be $n \times n$ matrices, and assume that B is invertible. Is it always true that $\operatorname{rank}(AB) = \operatorname{rank}(A)$? (Note: If your answer is *yes*, give a reason. If your answer is *no*, give a counter-example.)

Sol: The answer is yes.

Consider A, B, AB as linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ by multiplications to vectors. Since B is invertible, $B\vec{v}$ can be any vector in $\mathbb{R}^n$. Then

$$\begin{aligned} \operatorname{range}(AB) &= \{A(B\vec{v}) \mid \vec{v} \in \mathbb{R}^n\} \\ &= \{A\vec{w} \mid \vec{w} \in \mathbb{R}^n\} \\ &= \operatorname{range}(A). \end{aligned}$$

Count the dimensions. Get $\operatorname{rank}(AB) = \operatorname{rank}(A)$.

Rmk: There are many ways to do it.

For example, by matrix multiplication, have

$$\operatorname{Col}(AB) \subset \operatorname{Col}(A),$$

b/c columns of AB are lin. comb. of columns of A. Take dimensions, we have

$$\operatorname{rank}(AB) \leq \operatorname{rank}(A)$$

Similarly, have

$$\operatorname{rank}(\underline{AB} \cdot B^{-1}) \leq \operatorname{rank}(\underline{AB}) \quad (\text{think: } C = AB)$$