MATH 54 MIDTERM I, XINYI YUAN
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Problem Number	1	2	3	4	5	6	Total
Score							

YOUR NAME: ____________________

Last Name of Your Section Instructor (please circle): Agrawal, Dalal, Drouot, Gleason, Miller, Yott, Yuan, Zhang

Time of your discussion section: ____________

1. (5 points) Each of the following matrix is the augmented matrix of a system of linear equations. Write down the solution set of each system as a linear combination of vectors if it is consistent, explain the reason if it is inconsistent.

(a) 3-variable system:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

Sol: The system is

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

The solution set is the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(b) 3-variable system:

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Sol: The system is

$$\begin{cases} x_1 + 2x_3 = 0 \\ x_2 + 3x_3 = 0 \\ 0 = 1 \end{cases}$$

It is inconsistent.

(c) 4-variable system:

$$\begin{pmatrix} 1 & 0 & 2 & 3 & 4 \\ 0 & 1 & -5 & -6 & -7 \end{pmatrix}.$$

Sol: The system is

$$\begin{cases} x_1 + 2x_3 + 3x_4 = 4 \\ x_2 - 5x_3 - 6x_4 = -7 \end{cases}$$

A general solution is

$$\begin{cases} x_1 = -2x_3 - 3x_4 + 4 \\ x_2 = 5x_3 + 6x_4 - 7 \\ x_3 = \text{free} \\ x_4 = \text{free} \end{cases}$$

The solution set is

$$\begin{pmatrix} -2x_3 - 3x_4 + 4 \\ 5x_3 + 6x_4 - 7 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -2 \\ 5 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 6 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ -7 \\ 0 \\ 0 \end{pmatrix}$$

2. (5 points) Solve the system

$$\begin{cases} x_1 + 3x_2 + 5x_3 &= 1 \\ -x_1 + 2x_2 + 2x_3 &= 2 \\ x_1 + x_3 &= 3 \end{cases}.$$

(The final result could be either inconsistent or an expression of a general solution.)

Sol: Perform row operations on the augmented matrix:

$$\begin{pmatrix} 1 & 3 & 5 & 1 \\ -1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 3 \end{pmatrix} \xrightarrow{\text{move (R3) to top}} \begin{pmatrix} 1 & 0 & 1 & 3 \\ 1 & 3 & 5 & 1 \\ -1 & 2 & 2 & 2 \end{pmatrix}$$

$$\xrightarrow[\text{(R3)+(R1)}]{\text{(R2)-(R1)}} \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 3 & 4 & -2 \\ 0 & 2 & 3 & 5 \end{pmatrix} \xrightarrow{\text{(R2)-(R3)}} \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -7 \\ 0 & 2 & 3 & 5 \end{pmatrix}$$

$$\xrightarrow{\text{(R3)}-2\cdot\text{(R2)}} \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -7 \\ 0 & 0 & +1 & 19 \end{pmatrix} \xrightarrow[\text{(R2)-(R3)}]{\text{(R1)-(R3)}} \begin{pmatrix} 1 & 0 & 0 & -16 \\ 0 & 1 & 0 & -26 \\ 0 & 0 & 1 & 19 \end{pmatrix}$$

The system has a unique solution

$$\begin{cases} x_1 = -16 \\ x_2 = -26 \\ x_3 = 19 \end{cases}.$$

3. (5 points) Compute the inverse of the matrix

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 0 \end{pmatrix}.$$

Sol: Perform row operations on

$$\begin{pmatrix} 3 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Re-order the rows. Get

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Replace (R3) by (R3) $-3\cdot$(R1):

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -3 \end{pmatrix}.$$

Replace (R2) by (R2) $+ 2\cdot$(R3):

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & -6 \\ 0 & 0 & 1 & 1 & 0 & -3 \end{pmatrix}.$$

Therefore, the inverse is

$$\begin{pmatrix} 0 & 0 & 1 \\ 2 & 1 & -6 \\ 1 & 0 & -3 \end{pmatrix}.$$

4. (5 points) Let A be the matrix

$$A = \begin{pmatrix} 3 & 4 & a \\ 1 & 0 & 0 \\ 2 & 1 & 5 \end{pmatrix}.$$

Note that a is an entry of A. Find all values of a such that the linear transformation $L(\vec{x}) = A\vec{x}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ is not onto. (Note: "onto" and "surjective" have the same meaning.)

Sol: $L(\vec{x})$ is onto

$\Leftrightarrow$ $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$ in $\mathbb{R}^3$

$\Leftrightarrow$ A is invertible

$\Leftrightarrow$ $\det(A) \neq 0$.

Then the opposite:

$L(\vec{x})$ is not onto $\Leftrightarrow$ $\det(A) = 0$.

Expand $\det(A)$ by the second row. Get

$$\det(A) = -\det\begin{pmatrix} 4 & a \\ 1 & 5 \end{pmatrix} = -(20-a) = a-20.$$

Then $\det(A) = 0$ gives $a - 20 = 0$, $a = 20$.

5. (5 points) Let A, B, C be 3×3 matrices, and let $M = ABC$ be their product. Assume that the second row of A is $(1 \quad 2 \quad 3)$, the second column of C is $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$, and that

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

What is the $(2,2)$-entry of M? (Note: the $(2,2)$-entry means the entry in the second row and the second column.)

Sol. $M = A(BC)$

The (2,2)-entry of M is the dot product of

(2nd row of A) & (2nd column of BC)

By matrix multiplication, the (2nd column of BC) is

B · (2nd column of C).

Thus the (2,2)-entry of M is

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

$$= 4.$$

6. (5 points) Let
$$A = (\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3)$$
be a 3×3 matrix (so that $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are the columns of A.) Let
$$B = (\vec{a}_1 + \vec{a}_2 + \vec{a}_3 \quad \vec{a}_1 + 2\vec{a}_2 + 3\vec{a}_3 \quad \vec{a}_1 + 4\vec{a}_2 + 9\vec{a}_3)$$
be the 3×3 matrix whose columns are $\vec{a}_1 + \vec{a}_2 + \vec{a}_3$, $\vec{a}_1 + 2\vec{a}_2 + 3\vec{a}_3$, $\vec{a}_1 + 4\vec{a}_2 + 9\vec{a}_3$. Assume that $\det(A) = 2016$. Find $\det(B)$.

Sol: By matrix multiplication,

$$B = \left(A\begin{pmatrix}1\\1\\1\end{pmatrix} \quad A\begin{pmatrix}1\\2\\3\end{pmatrix} \quad A\begin{pmatrix}1\\4\\9\end{pmatrix} \right)$$

$$= A \begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 4 \\ 1 & 3 & 9\end{pmatrix}.$$

Then $\det(B) = \det(A) \cdot \det\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 4 \\ 1 & 3 & 9\end{pmatrix}$.

By row operations,

$$\begin{vmatrix}1 & 1 & 1\\ 1 & 2 & 4 \\ 1 & 3 & 9\end{vmatrix} = \begin{vmatrix}1 & 1 & 1\\ 0 & 1 & 3 \\ 0 & 2 & 8\end{vmatrix} = \begin{vmatrix}1 & 3 \\ 2 & 8\end{vmatrix} = 2.$$

Therefore,

$$\det(B) = 2016 \cdot 2 = 4032.$$

Rmk: Can also use column operations to convert B to A, and measure the change of the determinant. This gives a different solution.