Math 54 Final, Fall 2015

Name (Last, First): ______________________________

Student ID: ______________________________

GSI/Section: ______________________________

This is a closed book exam, no notes or calculators allowed. It consists of 9 problems, each worth 10 points. The lowest problem will be dropped, making the exam out of 80 points. Please avoid writing near the corner of the page where the exam is stapled, this area will be removed when the papers are scanned for grading. Put your extra work for each problem on the back of the page. If you run out of room on the back, use the pages attached at the back. If you want anything on those sheets graded, please indicate on the relevant problem which page your work is located on. **DO NOT REMOVE OR ADD ANY PAGES!**

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1. (10 points)

   (a) (2 points) State the rank theorem for an $m \times n$ matrix.

   (b) (3 points) Use the rank theorem to show that a system of 8 homogenous linear equations in 10 variables must have a nontrivial solution.

   (c) (5 points) Given an $n \times n$ matrix A, we have the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(x) = Ax$. Use the rank theorem to prove that if T is injective (one-to-one) then it must be surjective (onto). Conversely, prove that if T is surjective then it must be injective. In both cases, prove that T is an isomorphism.

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2. (10 points) A 3×3 *Jordan block* is a matrix of the form $J_c = \begin{bmatrix} c & 1 & 0 \\ 0 & c & 1 \\ 0 & 0 & c \end{bmatrix}$, where c is some constant.

   (a) (3 points) Find all eigenvalues of J_c, and calculate the corresponding eigenspaces.

   (b) (3 points) Prove that J_c is not diagonalizable.

   (c) (4 points) Prove that the matrix $A_c = \begin{bmatrix} c & 1 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$ is neither diagonalizable nor similar to J_c.

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3. (10 points) Mark the following as true or false. Justify your answers.

(a) (2 points) The Gram-Schmidt process can be used to turn a basis of eigenvectors into an orthogonal basis of eigenvectors.

(b) (2 points) The matrix $\begin{bmatrix} 1 & 10 & 100 \\ 10 & 20 & 30 \\ 100 & 30 & 40 \end{bmatrix}$ has only real eigenvalues.

(c) (2 points) If W is a subspace of an inner product space and $w_1, w_2, \ldots w_n$ is a basis of W, then the projection to W can be written

$$\mathrm{proj}_W(v) = \sum_{i=1}^{n} \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i.$$

(d) (2 points) The zero vector is orthogonal to every other vector.

(e) (2 points) $\langle f, g \rangle = f(0)g(0) + f(1)g(1)$ is an inner product on the vector space of continuous real-valued functions on the interval $[0, 1]$.

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4. (10 points) Let $M_{2\times 2}$ be the vector space of 2×2 matrices. Consider the linear transformation:

$$L : M_{2\times 2} \to M_{2\times 2},\ L(A) = A - A^T$$

(a) (3 points) Give a basis for $\mathrm{Ker}(L)$. Is L injective (one-to-one)?

(b) (2 points) Give a basis for the image of L.

Let $\mathbb{P}_2$ and $\mathbb{P}_3$ denote the vector space of polynomials of degree less than or equal to 2 or 3 respectively. Consider the linear transformation:

$$S : \mathbb{P}_2 \to \mathbb{P}_3,\ S(f(x)) = (x+1) \cdot f(x)$$

(c) (2 points) Is S injective (one-to-one)? Justify.

(d) (3 points) Is S surjective (onto)? Justify.

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5. (10 points) Find the general solution of the third-order differential equation $y''' - y'' + 4y' - 4y = e^t$.

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6. (10 points) Let $y_1(t)$ and $y_2(t)$ be real-valued differentiable functions on $(-\infty, \infty)$.

   (a) (2 points) Define what it means for $y_1(t)$ and $y_2(t)$ to be linearly independent on $(-\infty, \infty)$. Write your answer in complete sentences and be as precise as possible.

   (b) (2 points) Prove that t and e^t are linearly independent on $(-\infty, \infty)$.

   For parts *c*) and *d*), answer true or false. If the statement is true, prove it. If the statement is false, provide a counterexample.

   (c) (3 points) If $y_1(t)$ and $y_2(t)$ are linearly dependent, then $y_1'(t)$ and $y_2'(t)$ are linearly dependent.

   (d) (3 points) If $y_1(t)$ and $y_2(t)$ are linearly independent, then $y_1'(t)$ and $y_2'(t)$ are linearly independent.

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7. (10 points) Consider $y'' + 2by' + y = 0$ where b is a real parameter. For which values of b does this differential equation have a nontrivial solution $y(t)$ such that:

$$\lim_{t\to\infty} y(t) = 0.$$

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8. (10 points) (a) (7 points) Let $f(x) = |x|$ for $-\pi < x < \pi$. Compute the Fourier series of $f(x)$ on $(-\pi, \pi)$.

(b) (3 points) Use the previous part to compute the following sum:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

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9. (10 points) This problem concerns solutions to the heat equation:

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2}$$

with boundary conditions $u(0,t) = u(\pi,t) = 0$, and initial condition $u(x,0) = -5\sin(2x) + \sin(3x) = f(x)$. Recall that $u(x,t)$ measures the temperature at a point x and a time t on a rod of length π.

(a) (4 points) Using separation of variables $u(x,t) = X(x)T(t)$ as usual, we end up having to solve

$$X''(x) + \lambda X(x) = 0$$

with boundary conditions $X(0) = 0, X(\pi) = 0$. For which $\lambda \geq 0$ does this boundary value problem have a nonzero solution? Give the corresponding fundamental solutions to the heat equation (without initial condition).

(b) (2 points) Give the solution $u(x,t)$ to the heat equation above satisfying the initial condition $u(x,0) = f(x)$.

(c) (2 points) For any *fixed* value of x in the interval $(0,\pi)$, and for $u(x,t)$ as in the previous part, compute the following:

$$\lim_{t\to\infty} u(x,t)$$

(d) (2 points) Show $u(\frac{\pi}{2}, t)$ is increasing for $t > 0$.

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