Econ 101A – Final
Tu 16 December 2014

Problem 1. Utility maximization. (60 points)
In class, we considered the case of maximization of utility with two goods x and y. We also considered the case of intertemporal maximization of utility with consumption of goods today c_0 and in the future c_1. This problem asks you to combine these elements and reconsider some of the properties we found in the two separate cases. Consider a consumer with the utility function:

$$u(x_0, y_0, x_1, y_1) = x_0^{1/\alpha} + y_0^{1/\alpha} + \frac{1}{1+\delta}\left(x_1^{1/\alpha} + y_1^{1/\alpha}\right), \tag{1}$$

for $\alpha > 1$, x_t is the quantity of good x consumed at time $t = 0, 1$, and y_t is the quantity of good y_t consumed at time $t = 0, 1$.

1. What is the interpretation of δ in the utility function? Interpret the special cases of $\delta = 0$ and $\delta \to \infty$. (6 points)

2. Show that the utility function is concave in x_0 for $\alpha > 1$. (4 points)

3. Now consider the budget constraint. The price of good x is p_x and the price of good y is p_y in both periods. Denote income at time $t = 0$ as M_0 and at time $t = 1$ as M_1. The consumer can borrow across periods at interest rate r. Show that this implies the (intertemporal) budget constraint is:

$$p_x x_0 + p_y y_0 + \frac{1}{1+r}\left(p_x x_1 + p_y y_1\right) \leq M_0 + \frac{1}{1+r} M_1. \tag{2}$$

   (6 points)

4. The consumer maximizes utility with respect to the four consumption variables (x_0, y_0, x_1, y_1), subject to the budget constraint. Assume the budget constraint is satisfied with equality (it is) and write the Lagrangian function. Derive the set of first order conditions. (5 points)

5. Show that the first order conditions with respect to (x_0, y_0, x_1, y_1) lead to the following conditions:

$$\left(\frac{y_0}{x_0}\right)^{\frac{\alpha-1}{\alpha}} = \frac{p_x}{p_y} \tag{3}$$

   and

$$\left(\frac{y_1}{x_1}\right)^{\frac{\alpha-1}{\alpha}} = \frac{p_x}{p_y}. \tag{4}$$

   Give an economic interpretation of these conditions. Are you surprised that the same conditions that hold in the one-period utility maximization problem also hold in the 2-period one? (5 points)

6. Show that the first order conditions also imply that:

$$\left(\frac{y_1}{y_0}\right)^{\frac{\alpha-1}{\alpha}} = \left(\frac{x_1}{x_0}\right)^{\frac{\alpha-1}{\alpha}} = \frac{1+r}{1+\delta}. \tag{5}$$

   (7 points)

7. How does relative consumption in period 1 versus period 0 (i.e., y_1/y_0 and x_1/x_0) vary as r varies? Interpret. How does relative consumption in period 1 versus period 0 (i.e., y_1/y_0 and x_1/x_0) vary as δ varies? Interpret. (9 points)

8. What does equation (5) imply about y_1/y_0 when $r = \delta$? Why does the consumer smooth consumption across periods in this case? (6 points)

9. For $r = \delta$, use the budget constraint and equation (3) to derive an implicit function of the form $g(x_0, p_x, p_y, r, M_0, M_1) = 0$ [Note: your final expression should only involve x_0, p_x, p_y, r, M_0, and M_1.] (6 points)

10. Use the expression from point (9) and the implicit function theorem to compute $\partial x_0/\partial M_0$. Is good x at time $t = 0$ a normal good (for all levels of price and income)? (6 points)

Problem 2. General equlibrium. (40 points)
Consider a two-person, two-good exchange economy in which the utility functions are:

$$U_i(x_1^i, x_2^i) = x_1^i x_2^i, \quad i = 1, 2, \tag{6}$$

where x_1^i is the demand of consumer i for good 1 and x_2^i is the demand of consumer i for good 2. The initial endowments are $(\omega_1^1, \omega_2^1) = (1, 3)$ for consumer 1 and $(\omega_1^2, \omega_2^2) = (3, 1)$ for consumer 2.

1. Show that the set of Pareto optimal allocations can be expressed as

   $$\{(x_1^1, x_2^1) = (y, y),\ (x_1^2, x_2^2) = (4 - y, 4 - y),\ \text{for } y \in (0, 4)\}, \tag{7}$$

   i.e., the Pareto optimal allocations are such that each consumer consumes an equal amount of each good. Only consider strictly positive allocations. [Hint: At a Pareto optimal allocation the marginal rates of substitution (MRS) must be equal. Also note that for any allocation we must have $x_1^1 + x_1^2 = \omega_1^1 + \omega_1^2$ and $x_2^1 + x_2^2 = \omega_2^1 + \omega_2^2$.] (7 points)

2. Show that the contract curve can be expressed as

   $$\{(x_1^1, x_2^1) = (y, y),\ (x_1^2, x_2^2) = (4 - y, 4 - y),\ \text{for } y \in (\sqrt{3}, 4 - \sqrt{3})\}. \tag{8}$$

   [Hint: Recall that the individual rationality condition requires that $U_i(x_1^i, x_2^i) \geq U_i(\omega_1^i, \omega_2^i)$ for each consumer.] (7 points)

3. Draw the Edgeworth box for this economy, indicating the set of Pareto optimal allocations, the initial endowments, the set of allocations satisfying the individual rationality condition, and the contract curve. (7 points)

4. Show that any price vector $p = (p_1, p_2)$ where $p_1 = p_2$, along with the allocation $(x_1^1, x_2^1) = (x_1^2, x_2^2) = (2, 2)$, is a Walrasian equilibrium. [Recall there are two conditions that need to be satisfied in order to have a Walrasian equilibrium.] (7 points)

5. Are there any other equilibria? [Hint: write down the offer curves and find all points of intersection.] (5 points)

6. Now suppose that everything is the same, but we change the utility functions to

   $$U_i(x_1^i, x_2^i) = \min(x_1^i, x_2^i), \quad i = 1, 2. \tag{9}$$

   Draw the Edgeworth box for this economy, indicating the set of Pareto optimal allocations, the initial endowments, the set of allocations satisfying the individual rationality condition, and the contract curve. Write down an analytical expression for the contract curve, i.e., an expression like equation (8). (7 points)

Problem 3. Game theory. (40 points)

1. (a) Find all of the pure strategy and mixed strategy Nash equilibria of the following game:

$$\begin{array}{ccc} 1\backslash 2 & L & R \\ T & 6,0 & 0,6 \\ B & 3,2 & 6,0 \end{array} \tag{10}$$

   (b) Plot the best response functions of the two players and identify all of the Nash equilibria in your plot.

   (14 points)

2. (a) Find all of the pure strategy and mixed strategy Nash equilibria of the following game:

$$\begin{array}{ccc} 1\backslash 2 & L & R \\ T & 0,1 & 0,2 \\ B & 2,2 & 0,1 \end{array} \tag{11}$$

   (b) Plot the best response functions of the two players and identify all of the Nash equilibria in your plot.

   (Extra Credit, 14 points) (work on this last!)

3. **Approaching cars.** Members of a single population of car drivers are randomly matched in pairs when they simultaneously approach intersections from different directions. In each interaction, each driver can either stop or continue. The drivers' payoffs are given by:

$$\begin{array}{ccc} 1\backslash 2 & \mathit{Stop} & \mathit{Continue} \\ \mathit{Stop} & 1,1 & 1-\epsilon,2 \\ \mathit{Continue} & 2,1-\epsilon & 0,0 \end{array} \tag{12}$$

   The parameter ϵ, with $0 < \epsilon < 1$, reflects the fact that each driver dislikes being the only one to stop.

   (a) Find all of the pure strategy and mixed strategy Nash equilibria of this game.

   (b) Now suppose the drivers are (re)educated to feel guilty about choosing *Continue*. Assume their payoffs when choosing *Continue* fall by $\delta > 0$, so that the payoff matrix is now:

$$\begin{array}{ccc} 1\backslash 2 & \mathit{Stop} & \mathit{Continue} \\ \mathit{Stop} & 1,1 & 1-\epsilon,2-\delta \\ \mathit{Continue} & 2-\delta,1-\epsilon & -\delta,-\delta \end{array} \tag{13}$$

   Show that all drivers are *better off* in this game than they are in the original game. Why is society better off if everyone feels guilty about being aggressive?

   (12 points)

Problem 4. Worker effort and altruism. (60 points)
Consider the case of a firm with employees. The employee chooses the effort e. The firm pays the employees both a salary W and a piece-rate w times the units of effort e. Unlike in the moral hazard case we dealt with in class, in this case the effort is observable by the firm. The firm earns revenue pe for every unit e produced by the worker. The timing is such that the firm first sets the pay package (W, w) and then the worker chooses the optimal effort e. The worker has linear utility over money (that is, is risk-neutral) and thus gains utility $W + we$ from earnings at the workplace. In addition, the worker pays a cost of effort $ce^2/2$ (with $c > 0$) from exerting effort at the workplace. Hence the worker maximizes

$$\max_{e} \ W + we - ce^2/2$$

1. Solve for the optimal e^* of the worker as a function of W, w, and c. (7 points)

2. How does the optimal effort of the worker depend on the piece rate w? How about on the flat pay W? How about on the cost parameter c? Discuss the intuition. (7 points)

3. Consider now the problem of the firm. Remember that the firm revenue is pe. Thus the firm seeks to maximize

$$\begin{aligned} &\max_{W,w} \ pe - W - we \\ &\text{s.t. } e^* = e^*(W, w) \end{aligned}$$

   The firm chooses optimally the piece rate w and the flat-pay W subject to $w \geq 0$ and $W \geq 0$. Plug into the maximization problem the e^* you derived above. First, use a qualitative argument to find the optimal W^*. Then take first order conditions with respect to w. What is the solution? [If you have trouble here, just skip to the next point] (7 points)

4. Let's go back now to the employee problem and consider the case in which the worker is altruistic towards the employer by putting a weight α on the profits of the firm, with $0 < \alpha < 1$. In this case, the worker solves the problem

$$\max_{e} \ W + we - ce^2/2 + \alpha \left[pe - W - we\right].$$

   Derive the optimal effort e_α^* in this case. (7 points)

5. Now we compare the optimal effort with altruism e_α^* derived in point (4) for $0 < \alpha < 1$ to the optimal effort without altruism e^* derived in point (1). In doing the comparison, hold constant c, p, and also w, and W. Also, assume $p > w$. Which is larger, e_α^* or e^*? (Remember $0 < \alpha < 1$) Discuss the intuition. (7 points)

6. How do the two efforts e^* and e_α^* respond to an increase in the value of the product p? Compare the derivatives of e^* and e_α^* with respect to p . Discuss the intuition. (7 points)

7. How do the two efforts e^* and e^*_α respond to an increase in the value of the piece-rate w? Compare the derivative of e^* and e^*_α with respect to w and discuss which effort is more sensitive to the piece rate. Discuss the intuition. (7 points)

8. Consider now the case of a social planner who aims to maximize overall welfare in the economy. This particular social planner wants to maximize the sum of the utility of the employee and of the firm. Therefore, this planner maximizes

$$\left(W + we - ce^2/2\right) + \left(pe - W - we\right).$$

   Find the solution for e^*_{SP} which maximizes this expression. (6 points)

9. For which value of altruism α does the solution e^*_α equate the one set by the social planner? Explain the intuition. (5 points)