MATH 54 MIDTERM 1

February 18 2016, 12:40-2:00

Your Name	SOLUTIONS
Student ID	

Section number and leader	

Do not turn this page until you are instructed to do so.

Show all your work in this exam booklet. There are blank pages in between the problems for scratch work. **If you want anything on the extra pages to be graded, write "see next/previous page" and label your work there with the problem number.**
No material other than simple writing utensils may be used. *In the event of an emergency or fire alarm leave your exam (closed) on your seat and meet with your GSI outside.*
If you need to use the restroom, leave your exam with a GSI while out of the room.

Point values are indicated in brackets to the left of each problem. Partial credit will be given for explanations and documentation of your approach, even when you don't complete the calculation. In particular, if you recognize your result to be wrong (e.g. by checking!), stating this will yield extra credit.

When asked to explain/show/prove, you should make clear and unambiguous statements, using a combination of formulas and words or arrows. (The graders will disregard formulas whose meaning is not stated explicitly.)

[3] **1a)** Rewrite the system of linear equations

$$\begin{aligned} x_2 - x_4 &= 5 \\ x_1 + x_3 &= -1 \\ x_1 + x_2 + x_3 + x_4 &= 0 \end{aligned}$$

in terms of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, a vector $\mathbf{b}$ in $\mathbb{R}^p$, and an unknown $\mathbf{x}$ in $\mathbb{R}^q$. In particular, specify T and $\mathbf{b}$ explicitly. (This involves a choice of appropriate integers m, n, p, q.)

$T(\underline{x}) = \underline{b}$ with

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_2 - x_4 \\ x_1 + x_3 \\ x_1 + x_2 + x_3 + x_4 \end{bmatrix} \quad \text{or} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \underline{x} \quad , \quad \underline{b} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$$

[5] **1b)** Find a basis for the kernel of T. (That is, find linear independent vector(s) which span $\text{kernel}(T)$.)

$$T(\underline{x}) = \underline{0} \iff \begin{cases} x_2 - x_4 = 0 \\ x_1 + x_3 = 0 \\ x_1 + x_2 + x_3 + x_4 = 0 \end{cases} \iff \begin{cases} x_2 = x_4 \\ x_1 = -x_3 \\ \underbrace{-x_3 + x_4 + x_3 + x_4}_{2x_4} = 0 \end{cases} \iff \begin{cases} x_2 = x_4 = 0 \\ x_1 = -x_3 \end{cases}$$

$$\Rightarrow \text{kernel}(T) = \left\{ \begin{bmatrix} -x_3 \\ 0 \\ x_3 \\ 0 \end{bmatrix} \;\middle|\; x_3 \text{ any scalar} \right\} = \text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

linearly independent since $\neq \underline{0}$

[4] **1c)** Find the solution set for the system in 1a)

$$\begin{aligned} x_2 - x_4 &= 5 \\ x_1 + x_3 &= -1 \\ x_1 + x_2 + x_3 + x_4 &= 0 \end{aligned}$$

and write it in parametric vector form.
(You can use any method but should show your work or explain your reasoning.)

$$\left\{\begin{array}{l}\text{general solution} \\ \text{of } T(\underline{x}) = \underline{b}\end{array}\right\} = \text{particular solution} + \text{kernel}(T)$$

$$= \begin{bmatrix} -1 \\ 3 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

from 1b)

found by choosing $x_3 = 0$, then

$$\left\{\begin{array}{l} x_2 = 5 + x_4 \\ x_1 = -1 \\ -1 + (5 + x_4) + x_4 = 0 \end{array}\right. \iff \left\{\begin{array}{l} x_2 = 5 + (-2) = 3 \\ x_1 = -1 \\ x_4 = \frac{1-5}{2} = -2 \end{array}\right.$$

[6] **2)** Let $T : V \to W$ be a linear transformation between vector spaces V, W, and suppose that $w \neq 0$ in W is a vector for which $T(x) = w$ has exactly one solution x. What does this imply about $\mathrm{kernel}(T)$? Prove your statement without appealing to theorems or solution principles from book or lecture.

$$\underline{\underline{\mathrm{kernel}(T) = \{0\}}}$$

<u>Proof</u>: Let v be the solution of $T(x)=w$.

If $T(k)=0$

then $T(v+k) = T(v)+T(k) = w+0 = w$ by linearity

and hence $v+k = v$ by uniqueness of solutions of $T(x)=w$

$\Downarrow$

$k = -v+v = 0$

This shows $T(k)=0 \Leftrightarrow k=0$, so $\mathrm{kernel}(T) = \{0\}$.

[6] **3)** Let $\mathbb{P}$ be the vector space of polynomials and suppose that $T : \mathbb{P} \to \mathbb{R}^3$ is a linear transformation so that

$$T(1+t^2) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad T(t^2) = \begin{bmatrix} -1 \\ 0 \\ -4 \end{bmatrix}, \quad T(t^3) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Use this information to calculate $T(1+2t^2+3t^3)$.

(Hint: The functions $p_1(t) = 1+t^2$, $p_2(t) = t^2$, $p_3(t) = t^3$, $q(t) = 1+2t^2+3t^3$ above are vectors in $\mathbb{P}$. However, there is no need to use this notation.)

$$T(1+2t^2+3t^3) = T((1+t^2)+t^2+3\cdot t^3)$$

$$= T(1+t^2) + T(t^2) + 3T(t^3) \quad \text{by linearity}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ -4 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-1+9 \\ 2+6 \\ 3-4+3 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 2 \end{bmatrix}$$

(The first step uses $q = c_1 p_1 + c_2 p_2 + c_3 p_3$

$\iff 1+2t^2+3t^3 = c_1(1+t^2) + c_2 t^2 + c_3 t^3$ for all t

$\iff \begin{aligned} 1 &= c_1 \\ 2 &= c_1 + c_2 \\ 3 &= c_3 \end{aligned} \iff c_1 = c_2 = 1,\ c_3 = 3$)

[3] **4a)** Consider the functions $t^2 - 1$, t^3, and $\frac{1}{t^2+1}$ as vectors in $\mathcal{C}$, the vector space of continuous functions of $-\infty < t < \infty$. State the definition of what it would mean for these three vectors to be linearly independent in $\mathcal{C}$.

$$c_1 (t^2-1) + c_2 t^3 + c_3 \frac{1}{t^2+1} = 0 \quad \text{for all } t$$

only holds if $c_1 = c_2 = c_3 = 0$

[5] **4b)** Using your definition from 4a), show that $t^2 - 1$, t^3, and $\frac{1}{t^2+1}$ are linearly independent in $\mathcal{C}$.

(Hint: plug in convenient values of t.)

$$c_1 (t^2-1) + c_2 t^3 + c_3 \frac{1}{t^2+1} = 0 \quad \text{for all } t$$

$$\Downarrow$$

$$\begin{cases} t=0: & -c_1 - c_2 + c_3 = 0 \\ t=1: & c_2 + \frac{1}{2} c_3 = 0 \\ t=-1: & -c_2 + \frac{1}{2} c_3 = 0 \end{cases}$$

$$\left.\begin{matrix} c_2 + \frac{1}{2} c_3 = 0 \\ -c_2 + \frac{1}{2} c_3 = 0 \end{matrix}\right\} \Rightarrow \left(\tfrac{1}{2}+\tfrac{1}{2}\right) c_3 = 0$$

$$\Rightarrow \begin{cases} c_2 = \frac{1}{2} c_3 = 0 \\ c_1 = c_3 = 0 \end{cases}$$

$$\Downarrow$$

$$c_1 = c_2 = c_3 = 0$$

[5] **5)** Make a list of all possible reduced echelon forms of matrices A with 2 rows and 3 columns, so that the corresponding linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2, \mathbf{x} \mapsto A\mathbf{x}$ is onto.

$\Updownarrow$

pivot in each row of echelon form

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $*$ is any entry (including 0)

[3] **6a)** Give an example of two matrices $A \neq 0$ and $B \neq 0$ whose product is $AB = 0$.

many possibilities, e.g.

$$\begin{bmatrix}-1\\1\end{bmatrix} = B$$

$$A = \begin{bmatrix}1 & 1\end{bmatrix} \quad \rightarrow \quad AB = \begin{bmatrix}1\cdot(-1)+1\cdot 1\end{bmatrix} = \begin{bmatrix}0\end{bmatrix}$$

[5] **6b)** Someone claims they found an example of $AB = 0$ where A is invertible, and B is invertible, too. Prove them wrong!

$$\left.\begin{array}{l} A^{-1} \text{ exists} \\ AB = 0 \end{array}\right\} \Rightarrow B = IB = (A^{-1}A)B = A^{-1}(AB) = A^{-1}0 = 0$$

$$\left.\begin{array}{l} B^{-1} \text{ exists} \\ B = 0 \end{array}\right\} \Rightarrow I = B^{-1}B = B^{-1}0 = 0$$

$$\Rightarrow I = 0 \ ↯$$ The claim leads to contradictions, so must be wrong.

[5] **7)** Calculate the determinant of $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ 8 & 6 & 4 & 0 \\ 6 & 5 & 4 & 3 \end{bmatrix}$ by showing each step in a cofactor expansion.

$$\det(A) = 2 \det \begin{bmatrix} 5 & 0 & 0 \\ 6 & 4 & 0 \\ 5 & 4 & 3 \end{bmatrix} = 2 \cdot 5 \det \begin{bmatrix} 4 & 0 \\ 4 & 3 \end{bmatrix} = 2 \cdot 5 \, (4 \cdot 3 - 0 \cdot 4)$$

$$= 10 \cdot 12 = \underline{120}$$

[8] **8a)** For which parameter(s) h is $A = \begin{bmatrix} 6 & h & 3 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ invertible ?

For those parameter(s), find the inverse.

$$[A \mid I] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 6 & h & 3 & 1 & 0 & 0 \end{array}\right]$$

-6 $\quad -h \quad -\frac{1}{2}h \quad +6$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 3 & 1 & -h/2 & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 1/3 & -h/6 & 2 \end{array}\right]$$

REF of $A = I$ for all h $\Rightarrow$ <u>invertible for all h</u>

$$A^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1/2 & 0 \\ 1/3 & -h/6 & 2 \end{bmatrix}$$

[2] **8b)** What matrix multiplication would you compute to check your result in 8a)?
(Full credit is given for just writing down the matrices to multiply – but some calculation is recommended.)

$A^{-1}A = I$ or $AA^{-1} = I$

$$\begin{bmatrix} 6 & h & 3 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1/2 & 0 \\ 1/3 & -h/6 & 2 \end{bmatrix} \begin{bmatrix} (-1)(-1) & 0 & 0 \\ 0 & \frac{1}{2}\cdot 2 & 0 \\ \frac{6}{3}-2 & \frac{h}{3}-\frac{2h}{6} & \frac{1}{3}\cdot 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$