
1. A Modest Proposition: 3/3/3/3/3 Clearly indicate your correctly formatted answer: this is what is to be graded.No need to justify!

You have $Z = \forall x, (P(x) \implies (Q(x) \vee R(x)))$. State in each case below whether Z is certainly true, certainly false, or possibly true.

1. There is an x , such that $P(x)$ is true and either $Q(x)$ or $R(x)$ is true.

Z could be true.

The existence of an x is consistent with Z being true since it satisfies the “for all” statement but doesn’t guarantee that Z is true (because it was only shown for a particular value of x).

2. For every x , if $R(x)$ is false then $P(x)$ is false.

Z is always true.

If $R(x)$ is true the statement is true. If $R(x)$ is false, then $P(x)$ is false, which makes the implication vacuously true.

3. For every x where $(\neg Q(x) \wedge \neg R(x))$ is true, we have $P(x)$ is false. Z is proven.

This statement says that for every x , the contrapositive of the statement in Z is true, thus, Z is true.

4. For every x such that $Q(x)$ is false, then either $R(x)$ is true or $P(x)$ is false. Z is true.

This is a restatement of the statement in Z in or form.

5. There is an x such that $Q(x)$ is false and $R(x)$ is false and $P(x)$ is true. Z is false.

This is the negation of Z .

2. Short Answer/True/False/Maybe: 3/2/2/2/3/2/2/2/2 Clearly indicate your correctly formatted answer: this is what is to be graded.No need to justify!

1. Give a set of preferences, an unstable pairing for those preferences, and a rogue couple with respect to your pairing. (Again, we are asking for an *unstable* pairing.)

For men A and B :

A : 1,2

B : 2,1

For women:

1: A,B

2: B,A

An unstable pairing is $(A, 2)$ and $(B, 1)$. A rogue couple is $(A, 1)$.

2. (True/False.) If the preference lists of the men are all the same, the men’s least favorite woman is always paired with her least favorite man in the stable pairing returned by the traditional marriage algorithm.

False: it could happen that the man she ends up with, while not so popular with everyone else, is her favorite.

3. (True/False.) Any pairing where more than one man is matched to his least favorite partner is unstable.

False: Consider a two by two example where the men and women’s preference lists are mis-matched. The woman optimal pairing has both men being matched to their least favorite partner.

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4. (True/False.) In a run of the TMA (the traditional marriage algorithm), on one day, if a woman accidentally rejects a man she prefers to a man she keeps on the string, then the algorithm must terminate with a rogue couple.

False. Some intuition from the improvement lemma is that somebody better may come along, and she may well reject the guy anyway. No foul, no harm.

A specific example is the two by two instance where men A, B, prefer woman 1, and C prefers woman 2, woman 1 prefers C then A then B. On the first day, woman 1, rejects A accidentally, who then, on day 2, proposes to woman 2, who rejects C, who then asks woman 1. Woman 1 rejects B and B then, on day 3, proposes to woman 2. Woman 2 chooses her favorite of A and B, and finally one asks woman 3. For our example, it will be her favorite. In this example, all the woman end up with their favorite partner, and thus it is stable.

For the following questions. Consider a stable marriage instance on n men and n women: say one forms a graph consisting of vertices for each man and woman, and an edge for the *first* preference of each person. That is, if woman 1 prefers man A the most, there would be an edge $(1,A)$ in the resulting graph. Notice the graph may not be simple.

5. What is the maximum degree of the graph? (Short answer: an expression possibly involving n .)

$n + 1$

If a woman is everyone's favorite, her degree will be n due to their edges, and one more for her edge. This is the maximum due to the fact that the woman can participate in at most $n + 1$ edges; one corresponding to her preference and one for each man's first preference.

6. (True/False) If the maximum degree of the graph is two, the female optimal pairing pairs every female with her favorite partner (favorite means first on her preference list).

True.

Since the maximum degree of the graph is two, and there are $2n$ edges and $2n$ vertices, every vertex has degree 2 and the graph consists of disjoint cycles. Moreover, the graph is bipartite so the length of the cycles are even.

In any cycle, every other edge corresponds to a woman's favorite and forms a pairing of the elements of the cycle. Thus, the woman's favorites forms a pairing. It is clearly stable as no woman can be in a rogue pair with respect to this pairing as she is totally happy.

7. (True/False) If the maximum degree of the graph is two, there are only two stable pairings.

False.

Consider a 3 men/women example where each man's favorite lists that man as last, and each woman's favorite lists that woman as last.

The matching consisting of everyone's second favorite is stable, along with the matching consisting of the men's favorites, and the one consisting of the women's favorites.

That is, $A : (1, 2, 3), B : (2, 3, 1), C(3, 1, 2)$ and $1 : (B, C, A), 2 : (C, A, B), 3 : (A, B, C)$. The graph consists of edges $(A, 1), (1, B), (B, 2), (2, C), (C, 3), (3, A)$. The three stable pairings are the woman-optimal one $(1, B), (2, C), (3, A)$, the man-optimal one $(A, 1), (B, 2), (C, 3)$, and finally the middle one $(A, 2), (B, 1), (C, 3)$. The final one is stable since each person's favorite is the least favorite for the other person.

This ends the question on the first preference graph for stable marriage.

8. (True/False) If $a|b$ and $b|c$ then $a|c$. (Recall that $x|y$ means that x divides y .)

True: $b = ka$ and $c = lb$, so $c = kla$, for integers k, l .

9. How many edges need to be removed from a 3-dimensional hypercube to get a tree? (Short answer: a number.)

The 3 dimensional hypercube has $3(2^3)/2 = 12$ edges and 8 vertices. A tree on 8 vertices has 7 edges, so one needs to remove 5 edges.

3. More Short Answer: 3/3/3/3 Clearly indicate your correctly formatted answer: this is what is to be graded.No need to justify!

1. If $13x = 5 \pmod{46}$, what is x ? (Short answer.) Compute the inverse of 13 $\pmod{46}$ using iterative Euclid.

$$\begin{aligned}13(0) + 46(1) &= 46 \\13(1) + 46(0) &= 13 \\13(-3) + 46(1) &= 7 \\13(4) + 46(-1) &= 6 \\13(-7) + 46(2) &= 1\end{aligned}$$

This gives an inverse of -7, which says $x = -35 = 11 \pmod{46}$. Checking, we get $(13)(11) = 143 = (3) \times 46 + 5 = 5 \pmod{46}$.

2. What is the maximum number of solutions for x in the range $\{0, \dots, N-1\}$ for any equation of the form $ax = b \pmod{N}$, when $\gcd(a, N) = d$? (Short answer: an expression possibly involving N , a , b , and/or d .)

The maximum number of solutions is d .

If b is a multiple of d , we are looking for solutions to $ax = b + kN$ for integer k . But all of them are multiples of d , so we are looking for solutions to $a'x = b' \pmod{N'}$.

There is one solution to this equation modulo N' , since $\gcd(a', N) = 1$. Any solution of the form $x + iN'$ remains a solution, and there are d values of i where x remains in the range $\{0, \dots, N-1\}$.

3. What is $2^{50} \pmod{65}$? (Short answer: a number between 0 and 64 inclusive.) $65 = 13 \times 5$
We know that $a^{(p-1)(q-1)} = 1 \pmod{pq}$, so we have that $2^{48} = 1 \pmod{65}$ and that $2^{50} = 2^2 = 4 \pmod{65}$.

4. What is the size of the range of the function $f(x) = px \pmod{pq}$, where the domain is $\{1, \dots, pq-1\}$?
The range of a function is the set of values y , where $f(x) = y$ where x is in the domain. (Note: the set $\{0 \pmod{2}, 1 \pmod{2}, 2 \pmod{2}\}$ has size 2, since $0 = 2 \pmod{2}$.) (Short answer: an expression possibly using p and/or q .)

Every term is a multiple of p and there are q multiples of p between 0 and $pq-1$, using the definition that $pq = 0 \pmod{1}$. So we have q .

But actually, we made a mistake here, so we would take other answers such as $q-1$ as that would be the answer for $p=1$.

5. What is the smallest number of colors that can be used to properly color a tree? Recall that a proper coloring is an assignment of colors to vertices where for each edge (u, v) , u and v are assigned different colors. (Short answer.)

One can two color a tree by coloring a vertex, v , red and removing it and all the adjacent edges. Each adjacent neighbor is in a different connected component since a tree is acyclic.

Thus, we can two color (red and blue) the resulting components by induction. (The base case is a single vertex, which is trivial.) Then we make each of the neighbors blue (by switching the red and blue colors from the inductively assumed colorings if necessary). The edges in the components are two colored by induction and the fact that switching a pair of colors preserves two coloring, and the edge between v and its neighbors are two colored by construction: red for v and blue for each neighbor.

4. Some Proofs:3/6

1. Prove that for $x, y \in \mathbb{Z}$, that if $x - y > 536$, then $x > 268$ or $y < -268$.

By contrapositive. If $x \leq 268$ and $y \geq -268$, we have that $-y \leq 268$, and thus $x + (-y) \leq 536$. Thus, we have proven the contrapositive and thus the statement is true.

2. Show by induction that $\sum_{i=1}^n \frac{1}{i^3} \leq 2$.

We will prove the stronger theorem that $\sum_{i=1}^n \frac{1}{i^3} \leq 2 - \frac{1}{n^2}$ by induction.

Base case: For $n = 1$, $1 \leq 2 - \frac{1}{(1)^3} = 1$.

Induction hypothesis: $\sum_{i=1}^k \frac{1}{i^3} \leq 2 - \frac{1}{k^2}$.

$\sum_{i=1}^{k+1} \frac{1}{i^3} \leq (2 - \frac{1}{k^2}) + \frac{1}{(k+1)^3} = 2 - (\frac{1}{k^2} - \frac{1}{(k+1)^3})$

Now to complete the proof, we need to prove

$$\frac{1}{(k+1)^2} \leq \frac{1}{k^2} - \frac{1}{(k+1)^3}$$

Multiply by both sides by $(k+1)^2$, obtaining

$$\begin{aligned} 1 &\leq \frac{(k+1)^2}{k^2} - \frac{1}{k} \\ &\leq \left(\frac{k^2 + 2k + 1}{k^2} - \frac{1}{k} \right) \\ &\leq 1 + \left(\frac{2k + 1}{k^2} - \frac{1}{k} \right) \\ &\leq 1 + \left(\frac{2}{k} + \frac{1}{k^2} - \frac{1}{k} \right) \\ &\leq 1 + \left(\frac{1}{k} \right). \end{aligned}$$

Thus the inequality holds, and the statement follows.

5. Unique factorization. 4/3/4

In class, we proved that any number can be written as a product of primes. In this problem, you will prove that every number has a unique prime factorization. (**Warning: do not use the fact that the factorization is unique in this problem as the point is to prove this fact.**)

1. Prove that for a prime p that if $p|ab$ then $p|a$ or $p|b$. (You may use the fact that if $\gcd(x, y) = 1$ that there are integers m and n where $mx + ny = 1$.)

All the variables refer to integers in these solutions.

If $p \nmid a$ then $\gcd(p, a) = 1$ since p is prime and its only divisors are 1 and p .

Thus, we have $mp + na = 1$. Multiply both sides by b yields $bmp + nab = b$. We know that $kp = ab$ by the assumption that $p|ab$, so we get that $bmp + nkp = b$ or $p(bm + nk) = b$ and since b, m, k and n are integers we have that $p|b$.

2. Prove that if p is prime and $p|p_1 \cdot p_2 \cdots p_k$ that there is some i where $p|p_i$. (You may use part 1)

By induction. It is clearly true for $p_1 p_2$ by part a and for p_1 trivially. We assume the statement is true for the product of $k - 1$ integers. For k , we have that $p|p_1 b$ where $b = p_2 \cdot p_3 \cdots p_k$, and thus $p|p_1$ or $p|b$ by the previous statement. So either $p|p_1$ or by the induction hypothesis $p|p_i$ for some $i > 1$. \square

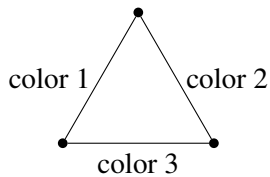
3. Prove that any natural number, $x \geq 2$ has a unique prime factorization; there is a unique multiset of primes whose product is x . For example, 12 is the product of the multiset $\{2, 2, 3\}$. And the multiset $\{2, 3, 2\}$ is the same multiset as $\{2, 2, 3\}$ but different from the multiset $\{2, 3\}$. Perhaps view them as sorted. (You may use the result from part 2.)

We prove the inductive statement on the value of x . Base case is for $x = 2$.

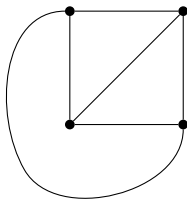
Assume that x has two factorizations $p_1 \dots p_m$ and $q_1 \dots q_n$. We can conclude from part 2 that $p_1 | q_i$ for some i . Thus, $p_1 = q_i$ for some i . Now by the (strong) inductive hypothesis that $\frac{x}{p_1}$ has a unique factorization, thus the multiset $\{p_2, \dots, p_m\}$ is the same as the multiset $\{q_1, \dots, q_{i-1}, q_{i+1}, q_n\}$ which implies that the original factorizations were also the same. \square

6. Edge Colorings: 3/3/3/3/4

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors.



1. (Short Answer) Show that the 4 vertex complete graph below can be 3 edge colored (use the numbers 1, 2, 3 for colors.)



Three color a triangle. Add the fourth vertex, notice that each edge has a different color available from the set of three colors. Done.

2. (Short Answer) How many colors are required to edge color a 3 dimensional hypercube?
3. Recall that edges connect vertices that differ in a dimension. And each vertex is incident to exactly one edge for each dimension. Thus, the entire set of edges for a specific dimension can be colored with a single color.
3. Prove that the complete graph on n vertices, K_n , can always be edge colored with n colors. (Hint: is $x + 1 \pmod{n}$ a bijection?)

Number the vertices: $\{0, \dots, n-1\}$, and the colors the same. For edge (i, j) , color it by $(i+j) \pmod{n}$ where we use the representative in $\{0, \dots, n-1\}$. For any two edges adjacent to vertex i , (i, j) and (i, j') we have that $i+j \neq i+j' \pmod{n}$, thus the coloring is legal.

Here, we used that i has an additive inverse and thus the function $f(x) = x + i \pmod{n}$ is a bijection.

4. Prove that any graph with maximum degree d can be edge colored with $2d - 1$ colors.

By induction. We will use a set of $2d - 1$ colors. Remove an edge and $2d - 1$ color the remaining graph from our set. This can be done by the induction hypothesis as the remaining graph's degree is no bigger than d and the graph has fewer edges. The edge is incident to two vertices each of which is incident to at most $d - 1$ other edges, and thus at most $2(d - 1) = 2d - 2$ colors are unavailable for edge e . Thus, we can color edge e without any conflicts.

5. Show that any tree has a degree 1 vertex. (You may use any definition of a tree that we provided in the notes, homeworks or lectures to prove this fact.)

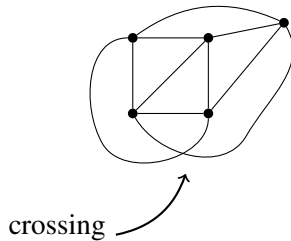
The number of edges in an n -vertex tree is $n - 1$, so the total degree is $2(n - 1)$, and the average degree as it most $2 - \frac{2}{n}$, thus there must be a vertex of degree at most 1.

6. Show that a tree can be edge colored with d colors where d is the maximum degree of any vertex.

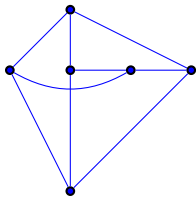
By induction. Base case is a single vertex, which has no edges to color, and thus can be colored with 0 colors. Remove the degree 1 vertex, v . Color the remaining tree with d colors. Note that vertex v 's neighboring vertex has degree at most $d - 1$ without the edge to v and thus its incident edges use at most $d - 1$ colors. Thus, there is a color available for coloring the edge incident to this vertex.

7. Planar Graphs:3/4/4

K_5 can be drawn in the plane with exactly one crossing as follows.



1. Draw $K_{3,3}$, the complete bipartite graph with three vertices on each side, in the plane where there is exactly one crossing.



2. Prove that K_6 cannot have a drawing in the plane with at most one crossing. (You may use the fact that for any planar graph with e edges and v vertices that $e \leq 3v - 6$.)

Assume a drawing of K_6 , which has 15 edges and 6 vertices, with one crossing, remove an edge, which leaves a planar drawing with 14 edges. But, Euler's formula suggest $e \leq 3v - 6$, or that $14 \leq 3(6) - 6 = 12$, which is ridiculous. Thus, K_6 does not have a planar drawing with one crossing.

Another method is to see that a crossing can be replaced by a vertex, which also breaks two edges into four. Thus, for a graph with a single crossing, one can obtain a planar graph with $e + 2$ edges and $v + 1$ vertices. Now K_6 has 15 edges and 6 vertices.

A planar drawing of K_6 would yield a planar graph with 17 edges and 7 vertices.

But by Euler's formula there is no such planar graph since $17 \not\leq 3(7) - 6 = 15$.

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3. Prove that for any planar graph where every cycle has length at least 6, there is a vertex of degree at most 2. (You may use Euler's formula: that $v + f = e + 2$ for any planar drawing with f faces of a graph with e edges and v vertices.)

Euler's formula is $v + f = e + 2$.

In a planar drawing, each edge is adjacent to at most 2 faces. And for a graph where the minimum length cycle is 6, each face is adjacent to at least 6 edges.

Thus, we have that $6f \leq 2e$.

Plugging in to Euler's formula, we see that $v + \frac{2}{6}e \geq e + 2$, or that $2e \leq 3v - 6$. Thus, the average degree, $\frac{2e}{v} \leq 3 - \frac{2}{v}$. \square