Physics 137A: Quantum Mechanics I, Spring 2006

Midterm I 2/23/06

Directions: The allotted time is 80 minutes. The 4 problems count equally. Two sides of your own notes are allowed. No books or calculators are allowed, and please ask for help only if a question's meaning is unclear.

1. Consider the harmonic oscillator ground state wavefunction $\psi(x) = Ae^{-m\omega x^2/2\hbar}$ with $A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$, which makes this a normalized wavefunction.

For this problem, you may wish to use the Gaussian integrals

$$\begin{aligned} \int_{-\infty}^{\infty} dx\, e^{-ax^2} &= \sqrt{\frac{\pi}{a}} \\ \int_{-\infty}^{\infty} dx\, x^2 e^{-ax^2} &= \sqrt{\frac{\pi}{4a^3}}. \end{aligned} \tag{1}$$

(a) (5 points) Write an integral for the probability to find the particle between $x = b$ and $x = c$. You do not need to do the integral for this part of the problem.

(b) (5 points) Is this wavefunction an eigenstate of kinetic energy? Justify your answer.

(c) (15 points) What is the expected value of kinetic energy of a free particle with this wavefunction? Try at least to write an integral; do the integral if you can.

2. At $t = 0$, suppose that a state in the infinite square well with potential

$$V(x) = \begin{cases} \infty & \text{if } x < 0 \text{ or } x > a \\ 0 & \text{if } 0 \leq x \leq a \end{cases} \tag{2}$$

has the wavefunction

$$\psi(x, t = 0) = \frac{1}{\sqrt{2}}\psi_1(x) + \frac{1}{\sqrt{2}}\psi_2(x), \tag{3}$$

where the normalized eigenstates are $\psi_n = \sqrt{\frac{2}{a}}\sin(n\pi x/a)$.

(a) (10 points) Is $\langle (x - a/2) \rangle$ positive or negative in this state (that is, is the particle on the left side or the right side of the well)? You do not need to calculate the integral if you can give a convincing answer without it.

(b) (15 points) Find a later time T at which $\langle (x - a/2)\rangle(t = T) = -\langle (x - a/2)\rangle(t = 0)$, so that the particle has moved to the other side of the well.

3. (a) (8 points) Prove that the expected value of the momentum operator is constant in time, if the system starts in an energy eigenstate.

(b) (7 points) Suppose that $\psi(x)$ is a bound eigenstate of an even potential $V(x) = V(-x)$. Suppose that it is real and nondegenerate (nondegenerate means that any eigenstate at the same energy is equal to $\psi(x)$ up to an overall phase factor). Prove that the wavefunction $\psi(x)$ must be either odd or even.

(c) (10 points) Suppose that all the states of a given potential are bound. Prove that the expectation value of energy in any state (not necessarily an eigenstate) is greater than or equal to the energy of the lowest bound state: $\langle E \rangle \geq E_0$. Justify the steps in your proof with a few words.

4. (a) (10 points) Consider the potential $V(x) = V_0 e^x$ with $V_0 > 0$. Are there any bound states with $E > 0$? Are there any bound states with $E < 0$? Explain your answers.

(b) (15 points) Consider the potential $V(x) = ax^4 + bx^2 + c$, with $a > 0$ and $b < 0$. Sketch this potential. Estimate the ground state energy by expanding the potential to quadratic order around one of its lowest points. It may help to recall that the harmonic oscillator ground state energy is $E = \hbar\omega/2$, when the potential is $V = m\omega^2 x^2/2$.