Stat 134 Final Fall 2015

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Name:

SID:

Circle your discussion section:

$9-10 \qquad 10-11 \qquad 11-12 \qquad 12-1 \qquad 1-2 \qquad 2-3$

There are 8 questions worth a total of 80 points. Attempt all questions and show your working. Answer the questions in the space provided. Additional space is available at the final page. Two double sided sheets of notes are permitted. No calculators or other references are permitted.

	Score
Q 1	
Q 2	
Q 3	
Q 4	
Q 5	
Q 6	
Q 7	
Q 8	
Total	

Question 1

Let A and B be two events such that $\mathbb{P}[A] = \mathbb{P}[B] = \frac{2}{3}$ and $\mathbb{P}[A \cap B] = \frac{1}{3}$.

(a) [2 points] Find $\mathbb{P}[A \cup B]$

Let X be the indicator of the event A and Y be the indicator of the event B.

(b) [3 points] Find $\mathbb{E}[X + Y]$.

(c) [4 points] Find the variance of $X + Y$.

Solutions

(a) By the inclusion exclusion formula

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B] = 1.$$

(b)Since X and Y are indicators,

$$\mathbb{E}[X + Y] = \mathbb{P}[A] + \mathbb{P}[B] = \frac{4}{3}.$$

(c) Since

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y)$$

and

$$\mathrm{Cov}(X, Y) = \mathbb{E}XY - \mathbb{E}X\mathbb{E}Y = \mathbb{P}[AB] - \mathbb{P}[A]\mathbb{P}[B] = \frac{1}{3} - (\frac{2}{3})^2 = -\frac{1}{9}.$$

we have

$$\mathrm{Var}(X + Y) = \mathbb{P}[A](1 - \mathbb{P}[A]) + \mathbb{P}[B](1 - \mathbb{P}[B]) - 2 \cdot \frac{1}{9} = \frac{2}{9}.$$

Question 2

A bag contains 6 red balls and 4 green balls. They are drawn out one by one without replacement.

(a) [3 points] Find the probability that exactly one of the first three balls is red.

(b) [3 points] Find the conditional probability that the first ball is red given that the second is green.

(c) [3 points] Find the probability that the first green ball comes on the fourth draw.

Solutions

(a) This is sampling without replacement so

$$\mathbb{P}[1 \text{ red}] = \frac{\binom{6}{1}\binom{4}{2}}{\binom{10}{3}}$$

(b) By symmetry,

$$\mathbb{P}[\text{Draw 1 red} \mid \text{Draw 2 green}] = \mathbb{P}[\text{Draw 2 red} \mid \text{Draw 1 green}] = \frac{6}{9}$$

(c)

$$\begin{aligned}\mathbb{P}[\text{First green on draw 4}] &= \mathbb{P}[\text{Red on draw 1,2,3, green on draw 4}]\\ &= \frac{6}{10}\cdot\frac{5}{9}\cdot\frac{4}{8}\cdot\frac{4}{7} = \frac{6_3\cdot 4}{10_4}\end{aligned}$$

Question 3

In a rate 2 Poisson process, N_t is the number of arrivals up to time t and T_k is the time of the k-th arrival.

(a) [3 points] Find $\mathbb{E}[T_4]$.

(b) [3 points] Find $\mathbb{P}[N_2 = 3, N_4 = 5]$.

(c) [4 points] Find $\mathbb{P}[N_1 = 2 \mid T_4 = 6]$.

Solutions

(a) The distribution of T_4 is Gamma$(4, 2)$ so $\mathbb{E}[T_4] = 4/2 = 2$.

(b) We have

$$\begin{aligned}\mathbb{P}[N_2 = 3, N_4 = 5] &= \mathbb{P}[N(0,2) = 3, N(2,4) = 2] = \mathbb{P}[N(0,2) = 3]\mathbb{P}[N(2,4) = 2] \\ &= \mathbb{P}[\text{Pois}(4) = 3]\mathbb{P}[\text{Pois}(4) = 2] = \frac{4^3 e^{-4}}{3!}\frac{4^2 e^{-4}}{2!} = \frac{256e^{-8}}{3}\end{aligned}$$

(c) Conditional on $T_4 = 6$ we have 3 arrivals uniformly chosen in $(0, 6)$. Thus

$$\mathbb{P}[N_1 = 2 \mid T_4 = 6] = \mathbb{P}[\text{Bin}(3, \frac{1}{6}) = 2] = \binom{3}{1}\frac{5}{6}(\frac{1}{6})^2 = \frac{5}{72}.$$

Question 4

Daisy is having another party. She invites 100 adult friends and their children. Of the adult friends 60 have no children, 30 have one child and 10 have two children. Each adult attends independently with probability $\frac{8}{10}$ and brings their children if they come. Let X be the number of adults who attend, Y be the number of children who attend and $Z = X + Y$ the total number who attend.

(a) [3 points] Find the probability that exactly 80 adults attend.

(b) [3 points] Find $\mathbb{E}[Y \mid X]$.

(c) [4 points] Using a normal approximation, find approximately $\mathbb{P}[Z \geq 125]$.

Solutions

(a) The number of adults who attend is $\text{Bin}(100, \frac{8}{10})$ so

$$\mathbb{P}[X = 80] = \binom{100}{80} (\frac{8}{10})^{80} (\frac{2}{10})^{20}$$

(b) The conditional expectation $\mathbb{E}[Y \mid X = x]$ is the expected sum of x samples with replacement from the population of families. A randomly chosen adult has $(30+2\cdot10)/100 = \frac{1}{2}$ children so $\mathbb{E}[Y \mid X = x] = x/2$. Thus $\mathbb{E}[Y \mid X] = X/2$.

(c) Let Z_i be the number of people in families with i children who attend the party. Then $Z_0 = \text{Bin}(60, \frac{8}{10})$, $Z_1 = 2\text{Bin}(30, \frac{8}{10})$ and $Z_2 = 3\text{Bin}(10, \frac{8}{10})$. These are independent so,

$$\text{Var}(Z) = \text{Var}(Z_0) + \text{Var}(Z_1) + \text{Var}(Z_2) = 60\frac{8}{10}\frac{2}{10} + 2^2 \cdot 30\frac{8}{10}\frac{2}{10} + 3^2 \cdot 10\frac{8}{10}\frac{2}{10} = 43.2$$

We have that

$$\mathbb{E}[Z] = 60 \cdot \frac{8}{10} + 2 \cdot 30 \cdot \frac{8}{10} + 3 \cdot 10 \cdot \frac{8}{10} = 120.$$

Then by a normal approximation

$$\mathbb{P}[Z \geq 125] \approx \mathbb{P}[N(0,1) \geq \frac{125 - 120 - \frac{1}{2}}{\sqrt{43.2}}] = 1 - \Phi\left(\frac{4.5}{\sqrt{43.2}}\right).$$

Question 5 The random variable X has density $f_X(x) = 6x^5$ for $0 < x < 1$.

(a) [2 points] Find $\mathbb{E}X$.

(b) [3 points] Find $\mathbb{P}[\frac{1}{3} \le X \le \frac{2}{3}]$.

(c) [3 points] Find the density of X^2.

(d) [4 points] The random variable Y is uniform on $(0, 2)$ and is independent of X. Find the density of $X + Y$.

Solutions

(a)

$$\mathbb{E}X = \int_0^1 x f_X(x) dx = \int_0^1 6x^6 dx = \frac{6}{7}.$$

(b)

$$\mathbb{P}[\frac{1}{3} \le X \le \frac{2}{3}] = \int_{\frac{1}{3}}^{\frac{2}{3}} f_X(x) dx = \int_{\frac{1}{3}}^{\frac{2}{3}} 6x^5 dx = x^6 \mid_{\frac{1}{3}}^{\frac{2}{3}} = (\frac{2}{3})^6 - (\frac{1}{3})^6 = \frac{63}{729}$$

(c) Let $g(x) = x^2$ By the change of variable formula if $Z = g(X) = X^2$,

$$f_Z(x^2) = \frac{f_X(x)}{|g'(x)|} = \frac{6x^5}{2x} = 3x^4,$$

so $f_Z(x) = 3x^2$ for $0 < x < 1$.

(d) The marginal distribution of Y is $f_Y(y) = \frac{1}{2}$ for $0 < y < 2$ so joint distribution of X and Y is

$$f(x, y) = 3x^5 I(0 < x < 1, 0 < y < 2)$$

Then if $W = X + Y$ the range of W is $(0, 3)$ and

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f(x, w, x) dx \\ &= \int_{-\infty}^{\infty} 3x^5 I(0 < x < 1, 0 < w - x < 2) dx \\ &= \int_0^1 3x^5 I(0 < w - x < 2) dx \\ &= \int_0^1 3x^5 I(w - 2 < x < w) dx \end{aligned}$$

This then splits into 3 cases. First case $0 < w < 1$, then

$$f_W(w) = \int_0^w 3x^5 dx = \frac{1}{2} w^6.$$

Case 2 when $1 < w < 2$,

$$f_W(w) = \int_0^1 3x^5 dx = \frac{1}{2}$$

Case 3 when $2 < w < 3$

$$f_W(w) = \int_{w-2}^{1} 3x^5 dx = \frac{1}{2}(1 - (w-2)^6).$$

So the density of $X + Y$ is

$$f_W(w) = \begin{cases} \frac{1}{2}w^6 & 0 \le w \le 1 \\ \frac{1}{2} & 1 \le w \le 2 \\ \frac{1}{2}(1-(w-2)^6) & 2 \le w \le 3 \\ 0 & \text{otherwise.} \end{cases}$$

Question 6 Let X have density $f_X(x) = \frac{1}{2} + x$ for $0 < x < 1$. The conditional distribution of Y given $X = x$ is $\text{Unif}(0, x)$.

(a) [3 points] Find the joint density of X and Y.

(b) [3 points] Find the marginal density of Y.

(b) [4 points] Find $\mathbb{E}[X + Y]$.

Solutions

(a) We are given that the conditional density of Y given X is

$$f_{Y|X}(y \mid x) = \frac{1}{x}$$

for $0 < y < x$. Then the joint density is

$$f(x, y) = f_X(x) f_{Y|X}(y \mid x) = (\frac{1}{2} + x)\frac{1}{x} = 1 + \frac{1}{2x}$$

for $0 < y < x < 1$.

(b) The marginal of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_y^1 1 + \frac{1}{2x} dx = 1 - y - \frac{1}{2} \log y$$

(c) Since Y is uniform on $[0, X]$, conditional expectation of Y given X is

$$\mathbb{E}[Y \mid X] = X/2$$

so

$$\mathbb{E}[X + Y] = \mathbb{E}[\mathbb{E}[X + Y \mid X]] = \mathbb{E}[\frac{3}{2}X] = \frac{3}{2} \int_0^1 x(\frac{1}{2} + x) dx = \frac{7}{8}$$

Question 7 Two random variables X and Y are jointly normal with correlation $\frac{1}{2}$. Their marginal distributions are $X \sim N(1,1)$ and $Y \sim N(3,4)$

(a) [4 points] Find $\mathbb{P}[X + 2Y > 3]$.

(b) [3 points] Find $\mathbb{E}[X \mid Y = 6]$.

(c) [3 points] Show that X and $Y - X$ are independent.

Solutions

(a) We first find the distribution of $X + 2Y$ which is normal since X and Y is jointly normal.

$$\mathbb{E}[X + 2Y] = 1 + 2 \cdot 3 = 7$$

and

$$\begin{aligned}\mathrm{Var}(X + 2Y) &= \mathrm{Var}(X) + \mathrm{Var}(2Y) + 2\mathrm{Cov}(X, 2Y) \\ &= \mathrm{Var}(X) + 4\mathrm{Var}(Y) + 4\mathrm{Cov}(X, Y) \\ &= \mathrm{Var}(X) + 4\mathrm{Var}(Y) + 4\mathrm{Corr}(X, Y)\mathrm{SD}[X]\mathrm{SD}[Y] \\ &= 1 + 16 + 4 \cdot \frac{1}{2} \cdot 1 \cdot 2 = 21\end{aligned}$$

so $X + 2Y \sim N(7, 21)$. Then

$$\mathbb{P}[X + 2Y > 3] = \mathbb{P}[N(7, 21) > 3] = \mathbb{P}[N(0,1) > \frac{3-7}{\sqrt{21}}] = 1 - \Phi\left(\frac{-4}{\sqrt{21}}\right) = \Phi\left(\frac{4}{\sqrt{21}}\right)$$

(b) Writing X and Y in standard units we have that

$$X^* = \frac{X-1}{1}, \qquad Y^* = \frac{Y-3}{2}.$$

so

$$\mathbb{E}[X \mid Y = 6] = \mathbb{E}[X^* + 1 \mid Y^* = \frac{6-3}{2}] = 1 + \mathbb{E}[X^* \mid Y^* = \frac{3}{2}].$$

Since $\mathbb{E}[X^* \mid Y^* = \frac{3}{2}] = \frac{3}{2}\rho = \frac{3}{4}$, we have $\mathbb{E}[X \mid Y = 6] = \frac{7}{4}$.

(c)

Calculating the covariance we have

$$\mathrm{Cov}(X, Y - X) = \mathrm{Cov}(X, Y) + \mathrm{Cov}(X, -X) = \mathrm{SD}(X)\mathrm{SD}(Y)\mathrm{Corr}(X, Y) - \mathrm{Var}(X) = 0$$

Since X and $Y - X$ are uncorrelated and they are normally distributed they are also independent.

Question 8 Let T_k be the k-th arrival times of a rate 1 Poisson process. Let $X = T_1$ and $Y = T_3$.

(a) [3 points] Find the joint density of X and Y.

(b) [4 points] Find $\mathbb{E}[X \mid Y]$.

(c) [3 points] Find the covariance of X and Y.

Solutions

(a) If $X \in (x, x+dx)$ and $Y \in (y, y+dy)$ then there must be 0 arrivals in $(0,x)$ one in $[x, x+dx)$ one in $[x+dx, y)$ and one in $[y, y+dy)$ so

$$\begin{aligned}
&\mathbb{P}[X \in (x, x+dx), Y \in (y, y+dy)] \\
&= \mathbb{P}[N(0,x) = 0, N[x, x+dx) = 1, N[x+dx, y) = 1, N[y, y+dy) = 1] \\
&= e^{-x} \cdot dx \cdot (y-x)e^{-(y-x)} \cdot dy = (y-x)e^{-y} dx dy
\end{aligned}$$

and so the joint density is $f(x,y) = (y-x)e^{-y}$ for $0 < x < y$.

(b) The marginal distribution of Y is a Gamma$(3,1)$ so it has density $f_Y(y) = \frac{1}{2!} y^2 e^{-y}$. Hence the conditional distribution of X given Y is

$$f_{X|Y}(x \mid y) = \frac{(y-x)e^{-y}}{\frac{1}{2!} y^2 e^{-y}} = \frac{2(y-x)}{y^2}$$

for $x \in (0,y)$. Then

$$\mathbb{E}[X \mid Y = y] = \int x f_{X|Y}(x \mid y) dx = \int_0^y x \frac{2(y-x)}{y^2} = \frac{x^2 y - \frac{2}{3} x^2}{y^2} \Bigg|_0^y = \frac{1}{3} y$$

and hence $\mathbb{E}[X \mid Y] = \frac{1}{3} Y$.

(c) Since X and Y are respectively Gamma$(1,1)$ and Gamma$(3,1)$ we have $\mathbb{E}X = 1$ and $\mathbb{E}Y = 3$. Also

$$\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[XY \mid Y]] = \mathbb{E}[Y \mathbb{E}[X \mid Y]] = \mathbb{E}[\frac{1}{3} Y^2] = \frac{1}{3}(\mathbb{E}[Y]^2 + \mathrm{Var}(Y)) = \frac{1}{3}(9+3) = 4,$$

so

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 4 - 3 = 1.$$

Additional Space

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