CS 70 Discrete Mathematics and Probability Theory
Fall 2015 Rao Midterm 1

PRINT Your Name: ________________________________, ________________________________

(last) (first)

SIGN Your Name: ____________________________________

PRINT Your Student ID: ____________________________________

CIRCLE your exam room: 2050 VLSB A1 Hearst Annex 120 Latimer 145 Dwinelle 100 GPB OTHER

Name of the person sitting to your left: ____________________________________

Name of the person sitting to your right: ____________________________________

- After the exam starts, please write your student ID (or name) on every odd page (we will remove the staple when scanning your exam).
- We will not grade anything outside of the space provided for a problem unless we are clearly told in the space provided for the question to look elsewhere.
- On the short answer questions 1-5. You need only give the answer in the format requested (e.g., true/false, an expression, a statement.) We note that an expression may simply be a number or an expression with a relevant variable in it. **We will only grade the answers, and are unlikely to even look at any justifications or explanations.**
- On questions 6 and 7, do give arguments, proofs or clear descriptions as requested.
- You may consult one single-sided sheet of notes. Apart from that, you may not look at books, notes, etc. Calculators, phones, and computers are not permitted.
- There are 12 single sided pages (including a basically blank question) on the exam. Notify a proctor immediately if a page is missing.
- **You may, without proof, use theorems and facts that were proven in the notes and/or in lecture.**
- **You have 90 minutes: there are 40 parts on this exam.**
  - **Problems 1-5: 30 short answers total. No justification required!**
  - **Problem 6: 3 short proofs.**
  - **Problem 7: 2 examples, 3 short proofs, an algorithm, and its correctness.**

Do not turn this page until your instructor tells you to do so.

1. Short Answer: Logic

Clearly indicate your correctly formatted answer: this is what is to be graded.

For each question, please answer in the correct format. When an expression is asked for, it may simply be a number, or an expression involving variables in the problem statement, you have to figure out which is appropriate.

(a) Let the statement, $(\forall x \in R, \exists y \in R)\ G(x,y)$, be true for predicate $G(x,y)$ and R being the real numbers. Which of the following statements is certainly true, certainly false, or possibly true.

(i) $G(3,4)$

(ii) $(\forall x \in R) G(x,3)$

(iii) $(\exists y) G(3,y)$

(iv) $(\forall y) \neg G(3,y)$

(v) $(\exists x) G(x,4)$

(b) True or False?
$(\forall x)(\exists y)(P(x,y) \wedge \neg Q(x,y)) \equiv \neg(\exists x)(\forall y)(P(x,y) \implies Q(x,y))$

(c) True or False?
$(\exists x)((\forall y P(x,y)) \wedge (\forall z Q(x,z))) \equiv (\exists x)((\forall y)(P(x,y)) \wedge (\exists x)(\forall z) Q(x,z)$

(d) Give an expression using terms involving $\vee, \wedge$ and $\neg$ which is true if and only if exactly one of X, Y, and Z are true. (Just to remind you: $(X \wedge Y \wedge Z)$ means all three of X,Y,Z are true, $(X \vee Y \vee Z)$ means at least one of X,Y and Z is true.)

2. Short Answer: Proof and some arithmetic.

Clearly indicate your correctly formatted answer: this is what is to be graded.

(a) If d divides xy then d divides x or d divides y. (True or false.)

(b) If every prime that divides x also divides y and vice versa , then $x = y$. (True or false.)

(c) If $gcd(x,y) = gcd(y,z)$ then the set of common divisors of x and y is the same as the set of common divisors of y and z. (True or False)

3. **Short Answer: Stable Marriage**

Clearly indicate your correctly formatted answer: this is what is to be graded.

The following questions refer to stable marriage instances with n men and n women, answer True/False or provide an expression as requested.

(a) For $n = 2$, or any 2-men, 2 woman stable marriage instance, man A has the same optimal and pessimal woman. (True or False)

(b) In any stable marriage instance, in the pairing in the TMA there is some man who gets his favorite woman (the first women on his preference list.) (True or False.)

(c) In any stable marriage instance with n men and women, if every man has a different favorite woman, a different second favorite, a different third, and so on, and every woman has the same preference list, how many days does it take for TMA to finish? (Form of Answer: An expression that may contain n.)

(d) Consider a stable marriage instance with n men and n women, and where all men have the same preference list, and all women have different favorites, and different second men, and so on. How many days does the TMA take to finish? (Form of Answer: An expression that may contain n)

(e) It is possible for a stable pairing to have a man A and a woman 1 be paired if A is 1's least preferred choice and 1 is A's least preferred choice. (True or False)

(f) It is possible for a stable pairing to have two couples where each person is paired with their lowest possible choice. (True or False)

(g) If there is a pairing, P, that consists of only pairs from man and woman optimal pairings, then it must be stable. In other words, if every pair in P is a pair either in the man optimal or the woman optimal pairing then P is stable. (True or false.)

4. **Short Answer: Graphs**

Clearly indicate your correctly formatted answer: this is what is to be graded.

(a) Bob removed a degree 3 node in an n-vertex tree, how many connected components are in the resulting graph. (An expression that may contain n)

(b) Given an n-vertex tree, Bob added 10 edges to it, then Alice removed 5 edges and the resulting graph has 3 connected components. How many edges must be removed to remove all cycles in the resulting graph? (An expression that may contain n.)

(c) Give a gray code for 3-bit strings. (Recall, that a gray code is a sequence of the strings where adjacent elements differ by one. For example, the gray code of 2-bit strings is $00, 01, 11, 10$. Note the last string is considered adjacent to the first and 10 differs in one bit from 00. Answer should be sequence of three-bit strings: 8 in all.)

(d) For all $n \geq 3$, the complete graph on n vertices, K_n has more edges than the d-dimensional hypercube for $d = n$. (True or False)

(e) The complete graph with n vertices where p is an odd prime can have all its edges covered with x Rudrata cycles: a cycle where each vertex appears exactly once. What is the number, x, of such cycles in a cover? (Answer should be an expression that depends on n.)

(f) Give a set of Rudrata paths that covers the edges of K_5, the complete graph on 5 vertices. (Each path should be a sequence (or list) of edges in K_5.)

5. Short Answer: Modular Arithmetic

Clearly indicate your correctly formatted answer: this is what is to be graded.

(a) What is the multiplicative inverse of $3 \pmod 7$?

(b) What is the multiplicative inverse of $n-1$ modulo n? (An expression that may involve n. Simplicity matters.)

(c) What is the solution to the equation $3x = 6 \pmod{17}$? (A number in $\{0,\ldots,16\}$ or "No solution".)

(d) Let $R_0 = 0; R_1 = 2; R_n = 4R_{n-1} - 3R_{n-2}$ for $n \geq 2$. Is $R_n = 2 \pmod 3$ for $n \geq 1$? (True or False)

(e) Given that $extended-gcd(53, m) = (1, 7, -1)$, that is $(7)(53) + (-1)m = 1$, what is the solution to $53x + 3 = 10 \pmod m$? (Answer should be an expression that is interpreted $\pmod m$), and shouldn't consists of fractions.)

6. Simple proofs.

(a) Prove or disprove that for integers a, b, if $a + b \geq 1016$ that either a is at least 508 or b is at least 508.

(b) Prove or disprove that $\sqrt{8}$ is irrational.

(c) Let $R_0 = 0; R_1 = 2; R_n = 4R_{n-1} - 3R_{n-2}$ for $n \geq 2$.
Prove that $R_n = 3^n - 1$ for all $n \geq 0$.

7. Matchings.

In this problem, we are given a bipartite graph: $G=(L,R,E)$ where there are two sets of vertices, L and R, and $E \subseteq L \times R$, or each edge is incident to a vertex in L and a vertex in R. We also know that every vertex has degree *exactly* d.

We wish to partition the edges into d perfect matchings: a perfect matching is a set of edges where every vertex is incident to exactly one edge in the matching. Another view is that each vertex is matched to another vertex; similar to a pairing in stable marriage except that the pair must correspond to an edge in the graph. A matching is a set of edges where the number of edges incident to any vertex is at most 1 (as opposed to equal to 1 for a perfect matching.)

(a) Draw a 6 vertex example graph that for $d=2$ that meets the conditions above for an instance.

(b) Indicate two matchings in your graph that cover the edges.

(c) Prove that for any instance of this problem that $|L|=|R|$. (Remember every vertex has degree d for any instance.)

(d) Prove that the length of any cycle in an instance of this problem is even.

(e) Prove that you can partition the edges in a simple cycle in this graph into exactly two perfect matchings with respect to the vertices in the cycle.

(f) Assume d is a power of 2; $d = 2^k$ for some natural number k. Give an efficient algorithm to compute a partition of the edges into perfect matchings. (Note that trying all possible partitions is not efficient. The algorithm should not take exponential time.)

(g) Prove your algorithm from the previous part is correct.

Extra space for problem 7, if necessary.