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(a) (10 Points) The quirky instructor produces a sudden, single, clap in an otherwise silent and empty lecture hall. His clap is so loud and so short in duration that it can be modeled as an impulse—namely, $x(t) = \delta(t)$. Determine a reasonably simple expression for, and provide a well-labeled plot of, the corresponding output signal $y(t) = h(t)$, the impulse response of the system representing the room's acoustics.

Hint: Your expression for $h(t)$ should be left as an infinite sum.

The feedback system gives $\alpha[\delta(t) + \beta h(t-T)] = h(t)$.

So we have

$h(0) = \alpha[\delta(0) + \beta(-T)] = \alpha$

$h(T) = \alpha[\delta(T) + \beta h(0)] = \alpha^2 \beta$

$h(2T) = \alpha[\delta(2T) + \beta h(T)] = \alpha^3 \beta^2$

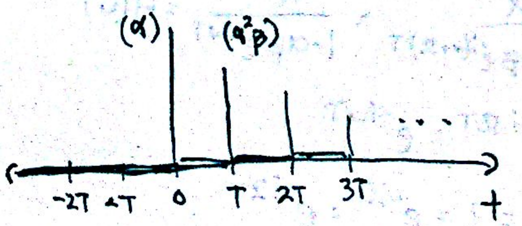
$\vdots$

$h(nT) = \alpha[\delta(nT) + \beta h(n-1)T] = \alpha^{n+1} \beta^n$

Also, we have $h(t) = 0,\ \forall t \neq kT,\ k \in \mathbb{Z}$.

Therefore,

$$h(t) = \sum_{n=0}^{+\infty} \alpha^{n+1} \beta^n \delta(t - nT)$$



(b) (5 Points) What constraint must the parameters α and β satisfy so that the system H is guaranteed to be BIBO stable? Provide a succinct, but clear and convincing, reasoning.

Since H is LTI, h must be absolutely integrable, so that H is BIBO stable.

$$\int_{-\infty}^{+\infty} |h(t)\, dt| < \infty \Rightarrow \int_{-\infty}^{\infty} \left| \sum_{n=0}^{+\infty} \alpha^{n+1} \beta^n \delta(t-nT) \right| dt < \infty$$

$$\Rightarrow \alpha \sum_{n=0}^{+\infty} (|\alpha\beta|)^n < \infty$$

$$\Rightarrow |\alpha\beta| < 1$$

Because $\alpha, \beta > 0$, $\boxed{\alpha\beta < 1}$

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(c) (30 Points) Assume in this part that α and β are such that the system H is BIBO stable. Determine a reasonably simple expression for $H(\omega)$, the frequency response of the system H that models the room's acoustics.

Show that despite being the frequency response of a continuous-time LTI system, $H(\omega)$ is periodic in ω, and determine the smallest $\Omega > 0$ such that $H(\omega + \Omega) = H(\omega)$ for all $\omega \in \mathbb{R}$.

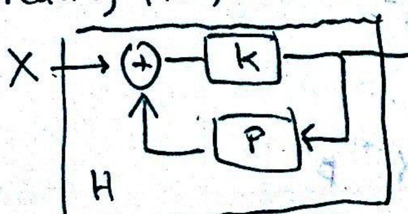
Then provide a well-labeled plot of the magnitude response $|H(\omega)|$.

Hint: You may want to first plot the magnitude response for the case $T = 1$, and from there infer the plot for the more generalize values of echo delay T.

You may or may not find it useful to note that

$$\sum_{\ell=0}^{\infty} \gamma^{\ell} = \frac{1}{1-\gamma}, \quad \text{if } |\gamma| < 1.$$

Treating H(w) like a feedback loop



We know $H(\omega) = \dfrac{K(\omega)}{1+K(\omega)P(\omega)}$

$K(\omega) = \alpha$

$P(\omega) = \beta G(\omega)$

$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t}\, dt$

$= \int_{-\infty}^{\infty} \delta(t-T) e^{-i\omega t}\, dt = e^{-i\omega T}$

$$\boxed{H(\omega) = \frac{\alpha}{1-\alpha\beta e^{-i\omega T}}}$$

Ω: $H(\omega+\Omega) = H(\omega)$

$H(\omega+\Omega) = \dfrac{\alpha}{1-\alpha\beta e^{-i(\omega+\Omega)T}} = \dfrac{\alpha}{1-\alpha\beta e^{-i\omega T}} = H(\omega)$

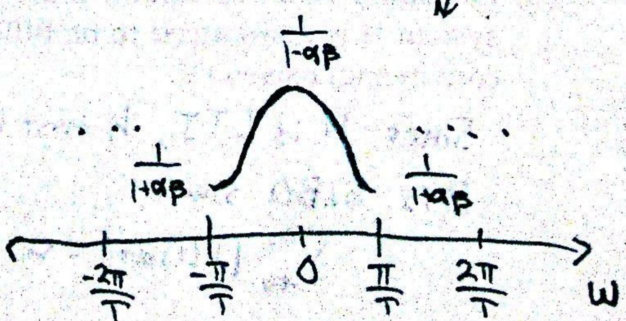
$\Rightarrow e^{-i\omega T} e^{-i\Omega T} = e^{-i\omega T}$

$e^{-i\Omega T} = 1 = e^{-i0} = e^{-i2\pi}$

$\Omega T = 2\pi$

$$\boxed{\Omega = \frac{2\pi}{T}}$$

Periodic



If not, find another signal that satisfies the property g...

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(d) (15 Points) Suppose the input signal is given by

$$\forall t \in \mathbb{R}, \quad x(t) = 1 + \cos\left(\frac{\pi t}{T}\right) + \sin\left(\frac{2\pi t}{T}\right).$$

$$X(t) = e^{i0t} + \frac{1}{2}e^{i\frac{\pi t}{T}} + \frac{1}{2}e^{-i\frac{\pi t}{T}} + \frac{1}{2i}e^{i\frac{2\pi t}{T}} - \frac{1}{2i}e^{-i\frac{2\pi t}{T}}$$

Determine a reasonably simple expression for the corresponding output $y(t)$.

$$y(t) = H(0)e^{i0t} + \frac{1}{2} H\left(\frac{\pi}{T}\right)e^{i\frac{\pi t}{T}} + \frac{1}{2}H\left(-\frac{\pi}{T}\right)e^{i\frac{2\pi t}{T}} + \frac{1}{2i}H\left(\frac{2\pi}{T}\right)e^{i\frac{2\pi t}{T}} - \frac{1}{2i}H\left(-\frac{2\pi}{T}\right)e^{-i\frac{2\pi t}{T}}$$

$$= \frac{1}{1-\alpha\beta} + \left(\frac{1}{1+\alpha\beta}\right)\left(\frac{1}{2}\right)\left(e^{i\frac{\pi t}{T}} + e^{-i\frac{\pi t}{T}}\right) + \left(\frac{1}{1-\alpha\beta}\right)\left(\frac{1}{2i}\right)\left(e^{i\frac{2\pi t}{T}} - e^{-i\frac{2\pi t}{T}}\right)$$

$$= \left(\frac{1}{1-\alpha\beta}\right)\left(1 - \sin\left(\frac{2\pi t}{T}\right)\right) + \left(\frac{1}{1+\alpha\beta}\right)\left(\cos\left(\frac{\pi t}{T}\right)\right)$$

Note: $H\left(\frac{\pi}{T}\right) = H\left(-\frac{\pi}{T}\right)$

$H(0) = H\left(\frac{2\pi}{T}\right) = H\left(-\frac{2\pi}{T}\right)$

(e) (5 Points) The quirkly instructor decides to increase the volume, thereby increasing α thereby increasing $\alpha\beta$ toward 1. What do you expect to hear at the threshold where $\alpha\beta = 1$.

Because the system is no longer stable, we should hear a horrific shrieking sound from the speaker. *

(f) (5 Points) The forward-path gain α is given to be constant over the range of frequencies of interest. However, if the system is BIBO stable—so that the frequency response is well-defined—is not constant. Explain qualitatively why this is the case. "Why is magnitude of freq. resp. not constant even though forward path gain is?"

The delay in the feedback path causes feedback and input to be out of phase with each other. This sometimes causes the sounds to be reinforced, but sometimes causes destructive feedback. Because of this relationship at different freq.; the magnitude of the freq. resp. is as well.

-We accepted any response which 5 demonstrated understanding given a reasonable interpretation of the question.

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MT2.2 (35 Points) Consider the discrete-time signal described by $x(n) = e^{i\pi n/3}$ for all integers n. This signal has fundamental period $p_x = 6$, and fundamental frequency $\omega_x = 2\pi/p_x = \pi/3$. We segment this signal into disjoint, contiguous intervals of duration $p = 3$ samples and we compute the length-3 DTFS expansion of each segment (corresponding to fundamental frequency $\omega_0 = 2\pi/3$).

In one or more parts of this problem, you may or may not find it useful to know the following facts:

$$\langle e^{i\pi n/3}, 1\rangle = \langle e^{i\pi n}, e^{i2\pi n/3}\rangle = \langle e^{-i\pi n/3}, e^{i4\pi n/3}\rangle = \sum_{n=0}^{2} e^{i\pi n/3} = \frac{4}{1 - i\sqrt{3}}$$

$$\langle e^{-i\pi n/3}, 1\rangle = \langle e^{i\pi n}, e^{i4\pi n/3}\rangle = \langle e^{i\pi n/3}, e^{i2\pi n/3}\rangle = \sum_{n=0}^{2} e^{i\pi n/3} = \frac{4}{1 + i\sqrt{3}}$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \qquad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} \qquad \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

(a) (15 Points) The signal x can be written as

$$x(n) = X_0 + X_1 e^{i\frac{2\pi}{3}n} + X_2 e^{i\frac{4\pi}{3}n}, \quad n = 0, 1, 2. \tag{1}$$

Determine a reasonably simple expression for each of the coefficients X_0, X_1, and X_2. You may continue your work in the blank space immediately above part (b) on the top of the next page.

Method 1 - DTFS Analysis

$p = 3$, $\omega_0 = \frac{2\pi}{3}$

represent $x[n]$ for $n = 0, 1, 2$

$X_k = \frac{1}{p}\sum_{\langle p\rangle} x[n] e^{-ik\omega_0 n} = \frac{1}{3}\sum_{n=0}^{2} e^{i\pi/3 n} e^{-ik\frac{2\pi}{3}n}$

$X_0 = \frac{1}{3}\sum_{n=0}^{2} e^{i\frac{\pi}{3}n} e^0 = \frac{1}{3}\left(\frac{1-e^{i\pi}}{1-e^{i\pi/3}}\right) = \frac{1}{3}\left(\frac{2}{\frac{1}{2}-\frac{\sqrt{3}}{2}i}\right)$

$= \frac{4}{3(1-i\sqrt{3})}$

$X_1 = \frac{1}{3}\sum_{n=0}^{2} e^{-i\pi/3 n} = \left(\frac{1}{3}\sum_{n=0}^{2} e^{i\pi/3 n}\right)^* = X_0^* = \frac{4}{3(1+i\sqrt{3})}$

$X_2 = \frac{1}{3}\sum_{n=0}^{2} e^{-i\pi n} = \frac{1}{3}\left((-1)^0 + (-1)^1 + (-1)^2\right)$

$= \frac{1}{3}$

Method 2 - Inner Products

$X_0 = \frac{\langle x[n], 1\rangle}{\langle 1, 1\rangle} = \frac{\langle e^{i\pi/3 n}, 1\rangle}{\langle 1, 1\rangle} = \frac{1}{3}\left(\frac{4}{(1-i\sqrt{3})}\right)$

$X_1 = \frac{\langle e^{i\pi/3 n}, e^{-i\frac{2\pi}{3}n}\rangle}{\langle e^{-i\pi/3 n}, e^{-i\frac{2\pi}{3}n}\rangle} = \frac{1}{3}\left(\frac{4}{1+i\sqrt{3}}\right)$

$X_2 = \frac{\langle e^{i\pi/3 n}, e^{-i\frac{4\pi}{3}n}\rangle}{\langle e^{-i4\pi/3 n}, e^{-i\frac{4\pi}{3}n}\rangle}$ ← not given, so have to use DTFS analysis or solve inner prod. by hand

$= \frac{1}{3}$

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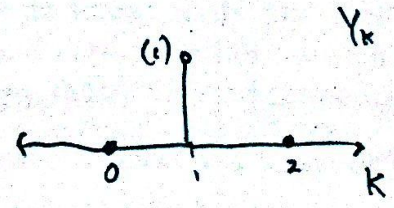
(b) (10 Points) Suppose we send the signal x through a nonlinear system whose output y is the pointwise (instantaneous) square of the input. That is, $y(n) = x^2(n)$ for all integers n. Determine, and provide a well-labeled plot of, Y_k, the length-3 DTFS coefficients for the output signal y.

$y[n] = (x[n])^2 = e^{i2\pi n/3}$ ⇐ Already written as linear combination of complex sinusoids and thus DTFS is readily known.

$$y[n] = \sum_{k=0}^{2} Y_k e^{ik2\pi n/3}$$

$$= 0 \times e^{i0\pi n/3} + 1 \times e^{i2\pi n/3} + 0 \times e^{i\frac{4\pi n}{3}}$$

$\Rightarrow Y_0 = 0, \ Y_1 = 1, \ Y_2 = 0$



(c) (10 Points) In part (a) you learned that *Spectral leakage* occurs if we take a length-3 DTFS of the signal x. That is, even though x has a single frequency—namely, $\omega_x = \pi/3$—every DTFS coefficient X_k becomes nonzero if the DTFS length p is not an integer multiple of the signal's fundamental period p_x. More generally, let's perform a length-p DTFS expansion of a periodic discrete-time signal x whose fundamental period is p_x. Suppose that p is *not* an integer multiple of p_x. Explain why *every* DTFS coefficient in the expansion must be nonzero.