Solutions

EECS 20, Section 2, Fall 2015

Midterm #1, September 24, 2015

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Problem 1

Choose one or more correct choices for each question. **Show all your reasoning and work.**

1. (3 Points) Let z_1 and z_2 be the complex numbers. $(z_1 + z_2)^*$ and $(z_1.z_2)^*$ are given by
   a. $z_1^* - z_2^*$ and $\frac{z_1^*}{z_2^*}$ respectively.
   b. $z_1^* + z_2^*$ and $z_1^*.z_2^*$ respectively.
   c. $z_1^* + z_2^*$ and $\frac{z_1^*}{z_2^*}$ respectively.
   d. None of the above.

   (Recall that z^* denotes the complex conjugate of z).

Let $z_1 = a + ib$, $z_2 = c + id \Rightarrow z_1^* = a - ib$, $z_2^* = c - id$

$(z_1 + z_2)^* = (a + c + i(b+d))^* = a + c - i(b+d) = z_1^* + z_2^*$

$(z_1 z_2)^* = (ac - bd + i(ad+bc))^* = ac - bd - i(ad+bc) = z_1^* z_2^*$

2. A discrete-time system F is defined for $n \in \mathbf{Z}$ as $F(x(n)) = x(n)$ for $n \geq 0$, and $F(x(n)) = -x(n)$ for $n < 0$.

   2.1 (3 Points) F is a linear system. True or False?

Let $y(n) = F(x(n))$. Consider $\hat{x}(n) = \alpha x(n)$.

Then $\hat{y}(n) = F(\hat{x}(n)) = F(\alpha x(n)) = \alpha x(n), n \geq 0; = -\alpha x(n), n < 0$

$= \alpha y(n) \Rightarrow$ Homogeneity

Let $y_1(n) = F(x_1(n))$, $y_2(n) = F(x_2(n))$. Consider $\hat{x}(n) = x_1(n) + x_2(n)$

then $\hat{y}(n) = F(\hat{x}(n)) = F(x_1(n) + x_2(n)) = x_1(n) + x_2(n), n \geq 0; = -(x_1(n) + x_2(n)), n < 0$

$= y_1(n) + y_2(n) \Rightarrow$ Additivity

   2.2 (3 Points) F is a time-invariant system. True or False?

Suppose $x(n) = 1 \quad \forall n$

Then, $y(n) = F(x(n)) = \begin{cases} 1 & n \geq 0 \\ -1 & n < 0 \end{cases}$

Consider shifting $x(n)$ to right by 1, i.e., $x(n-1) = \hat{x}(n)$

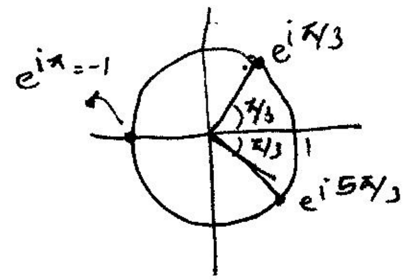
$\Rightarrow \hat{x}(n) = 1 \quad \forall n \Rightarrow \hat{y}(n) = y(n)$

which is $\neq y(n-1)$ (i.e. $y(n)$ shifted to right by 1)

3. (3 Points) Let $H(\omega) = |\omega|, -\infty < \omega < \infty$. We can find an LTI system for which $H(\omega)$ is the Frequency Response function. True or False.

$H(\omega)$ has to be periodic with period of 2π.

4. (3 Points) For the equation $z^3 = -1$,
   a. There is only one complex root at $z = -1$.
   b. There are triple complex roots at $z = -1$.
   c. There are three roots at $z = -1, e^{i\pi/3}$ *and* $e^{i5\pi/3}$.
   d. None of the above.



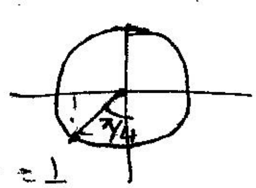
By the theorem stated in lecture, there are 3 roots for the equation $z^3 = -1$ or $z^3 + 1 = 0$

$$z^3 = -1 = e^{i(2k+1)\pi} \quad \forall k \in \mathbb{Z}$$

$$\Rightarrow z = \left(e^{i(2k+1)\pi}\right)^{1/3} = e^{i\left(\frac{2k+1}{3}\right)\pi} \quad \forall k \in \mathbb{Z}$$

For $k = 0, 1$ and 2, we get different roots as $e^{i\pi/3}$, $e^{i\pi}$, and $e^{i\frac{5\pi}{3}}$, respectively. Note that $e^{i\pi} = -1$.

5. (3 Points) $-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$ can be represents as
   a. $e^{i\pi/4}$
   b. $e^{i3\pi/4}$
   c. $e^{i5\pi/4}$
   d. $e^{i7\pi/4}$



Recall $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

6. For a system F, input $x(t)$ and output $y(t)$ are related as $y(t) = \alpha \int_{-\infty}^{t} x(\tau)d\tau$, where α is a constant.

6.1 (3 Points) F is a linear system. True or False?

$\hat{x}(t) = \beta x(t)$. Then $\hat{y}(t) = \alpha \int_{-\infty}^{t} \beta x(\tau)\, d\tau$

$= \beta \left(\alpha \int_{-\infty}^{t} x(\tau)\, d\tau\right) = \beta y(t)$

$\Rightarrow$ Homogeneity.

Let $y_1(t) = \alpha \int_{-\infty}^{t} x_1(\tau)d\tau$, $y_2(t) = \alpha \int_{-\infty}^{t} x_2(\tau)d\tau$

$\hat{x}(t) = x_1(t) + x_2(t)$. Then, $\hat{y}(t) = \alpha \int_{-\infty}^{t} (x_1(\tau) + x_2(\tau))\, d\tau = \alpha \int_{-\infty}^{t} x_1(\tau)d\tau + \alpha \int_{-\infty}^{t} x_2(\tau)d\tau$

$= y_1(t) + y_2(t)$

$\Rightarrow$ Additivity

6.2 (3 Points) F is a time-invariant system. True or False?

Let $\hat{x}(t) = x(t - T)$

then $\hat{y}(t) = \alpha \int_{-\infty}^{t} x(\tau - T)\, d\tau$. Let $u = \tau - T$

Then $\hat{y}(t) = \alpha \int_{-\infty}^{t-T} x(u)\, du$ (Note: $du = d\tau$, $\tau = -\infty \Rightarrow u = -\infty$, $\tau = t \Rightarrow u = t - T$)

$= y(t - T)$

7. (3 Points) Let z be a complex number, and θ be its phase. Then the phase of $z.i$ (i.e., z multiplied by the imaginary unit i) is given by
   a. $\theta + \frac{\pi}{4}$
   b. $\theta - \frac{\pi}{4}$
   c. $\theta + \frac{\pi}{2}$
   d. $\theta - \frac{\pi}{2}$

Let $z = Re^{i\theta}$

$i = e^{i\pi/2}$

$\Rightarrow z \cdot i = Re^{i\theta} \cdot e^{i\pi/2}$

$= Re^{i(\theta + \pi/2)}$

$\Rightarrow \angle z \cdot i = \theta + \frac{\pi}{2}$

8. (3 Points) Observe ~~that~~ for the complex discrete time signal $x(n) = e^{i\pi n}, n \in \mathbf{Z}$, the angular frequency is π radians/sec. The corresponding frequency in cycles/sec is

a. ¼
b. ½
c. 1
d. 2

$\omega = 2\pi f$ or $f = \frac{\omega}{2\pi}$ cycles/sec.

$\omega = \pi \Rightarrow f = \frac{1}{2}$ cycles/sec.

Problem 2

Consider an LTI system with the Impulse Response function $h(n), n \in \mathbf{Z}$ and the Frequency Response function $H(\omega), -\pi \le \omega \le \pi$. Assume that $h(n)$ is a real number for each n.

1. (10 Points) Prove that $H(-\omega) = H^*(\omega), -\pi \le \omega \le \pi$, where * denotes complex conjugate.

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)\, e^{-i\omega k}$$

$$H(-\omega) = \sum_{k=-\infty}^{\infty} h(k)\, e^{i\omega k}$$

$$H^*(\omega) = \left(\sum_{k=-\infty}^{\infty} h(k)\, e^{-i\omega k}\right)^* = \sum_{k=-\infty}^{\infty} \left(h(k)\cdot e^{-i\omega k}\right)^*$$

(since conjugate of sum = sum of conjugates)

$$= \sum_{k=-\infty}^{\infty} h^*(k)\cdot\left(e^{-i\omega k}\right)^*$$

(since conjugate of product is product of conjugate.)

$$= \sum_{k=-\infty}^{\infty} h(k)\, e^{i\omega k} = H(-\omega)$$

Since $h(k)$'s are real & $(e^{-i\omega k})^* = e^{i\omega k}$

Now **for all the following parts of Problem 2**, suppose $h(n) = \delta(n) - 3\delta(n-2), n \in \mathbf{Z}$.

2. (10 Points) For the input signal is $x(n) = 2\delta(n) + \delta(n-2), n \in \mathbf{Z}$, find and sketch the corresponding output signal $y(n), n \in \mathbf{Z}$.

For a general $x(n)$, output will be $h * x(n)$

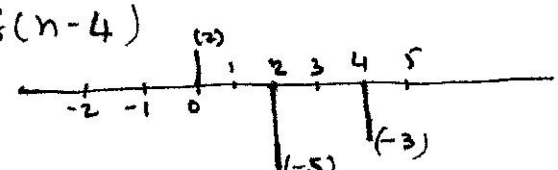
$$= \sum_{k=-\infty}^{\infty} h(k)\cdot x(n-k) = \sum_{k=-\infty}^{\infty} \left(\delta(k) - 3\delta(k-2)\right)\cdot x(n-k)$$

$$= x(n) - 3x(n-2)$$

Hence, for $x(n) = 2\delta(n) + \delta(n-2)$

$$y(n) = 2\delta(n) + \delta(n-2) - 3\left(2\delta(n-2) + \delta(n-4)\right)$$

$$= 2\delta(n) - 5\delta(n-2) - 3\delta(n-4)$$



3. (10 Points) Find the Frequency Response $H(\omega), -\pi \le \omega \le \pi$.

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k) e^{-i\omega k}$$

$$= \sum_{k=-\infty}^{\infty} (\delta(k) - 3\delta(k-2)) e^{-i\omega k}$$

$$= 1 - 3e^{-i2\omega}$$

4. (5 Points) Find the Magnitude Response $|H(\omega)|, -\pi \le \omega \le \pi$.

$$|H(\omega)| = |1 - 3e^{-i2\omega}|$$

$$= |1 - 3(\cos(-2\omega) + i\sin(-2\omega))|$$

(by Euler's Formula)

$$= |(1 - 3\cos(2\omega)) + i3\sin(2\omega)|$$

(since $\cos(-\theta) = \cos\theta$ & $\sin(-\theta) = -\sin\theta$)

$$= \sqrt{(1 - 3\cos(2\omega))^2 + 9\sin^2(2\omega)}$$

$$= \sqrt{10 - 6\cos(2\omega)}$$

(since $\sin^2(2\omega) + \cos^2(2\omega) = 1$)

5. (5 Points) Find the Phase Response $\angle H(\omega), -\pi \leq \omega \leq \pi$.

$$\angle H(\omega) = \angle\Big(\big(1 - 3\cos(2\omega)\big) + i\,3\sin(2\omega)\Big)$$

$$= \tan^{-1} \frac{3\sin(2\omega)}{1 - 3\cos(2\omega)}$$

6. (5 Points) For the input signal $x(n) = (-1)^n, n \in \mathbf{Z}$, find the corresponding output signal $y(n), n \in \mathbf{Z}$ using only $H(\omega)$ and $x(n)$.

$$x(n) = (-1)^n = e^{i\pi n}, \quad n \in \mathbb{Z}$$

Here $\omega = \pi$

$$\Rightarrow \quad y(n) = H(\pi) \cdot e^{i\pi n}$$

$$= \left(1 - 3e^{-i2\pi}\right) \cdot e^{i\pi n}$$

$$= -2e^{i\pi n} = -2(-1)^n, \quad n \in \mathbb{Z}$$

$\hookrightarrow$ (since $e^{-i2\pi} = 1$)

$$\Rightarrow \quad y(n) = \begin{cases} -2 & n \text{ even} \\ 2 & n \text{ odd} \end{cases} \quad (n \in \mathbb{Z})$$

7. (15 Points) For the input signal $x(n) = \cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right), n \in \mathbf{Z}$, show that the output signal $y(n)$ is of the form $\alpha.\cos(\beta n + \gamma)$, where α, β and γ are some real numbers. Also, find the values of α, β and γ. (Hints: $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$; $z + z^* = 2.Re(z)$ where z^* is complex conjugate of z, and $Re(z)$ denotes the real part of z.)

$$x(n) = \cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right) = \frac{e^{i\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)} + e^{-i\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)}}{2}$$

$$= \frac{1}{2}e^{i\frac{\pi}{4}} \cdot e^{i\frac{n\pi}{2}} + \frac{1}{2}e^{-i\frac{\pi}{4}} \cdot e^{-i\frac{n\pi}{2}}$$

Observe that $e^{i\frac{n\pi}{2}}$ as input gives output of $H\left(\frac{\pi}{2}\right) \cdot e^{i\frac{n\pi}{2}}$

& $e^{-i\frac{n\pi}{2}}$ " " " " $H\left(-\frac{\pi}{2}\right) \cdot e^{-i\frac{n\pi}{2}}$

Due to the linearity of the system, we have

$$y(n) = \frac{1}{2}e^{i\frac{\pi}{4}} \cdot H\left(\frac{\pi}{2}\right)e^{i\frac{n\pi}{2}} + \frac{1}{2}e^{-i\frac{\pi}{4}} \cdot H\left(-\frac{\pi}{2}\right)e^{-i\frac{n\pi}{2}}$$

From Problem 2-1, we know that $H\left(-\frac{\pi}{2}\right) = H^*\left(\frac{\pi}{2}\right)$

Hence, $y(n) = \frac{1}{2}e^{i\frac{\pi}{4}} H\left(\frac{\pi}{2}\right)e^{i\frac{n\pi}{2}} + \frac{1}{2}e^{-i\frac{\pi}{4}} H^*\left(\frac{\pi}{2}\right) \cdot e^{-i\frac{n\pi}{2}}$

Let $A = \frac{1}{2}e^{i\frac{\pi}{4}} H\left(\frac{\pi}{2}\right) e^{i\frac{n\pi}{2}}$. Then,

$y(n) = A + A^*$ (since conjugate of product = product of conjugates)

$$= 2\,Re(A)$$

$$= 2\,Re\left(\frac{1}{2}e^{i\frac{\pi}{4}}\underbrace{\left(1-3e^{-i2\cdot\frac{\pi}{2}}\right)}_{=4} \cdot e^{i\frac{n\pi}{2}}\right)$$

$$= 4\,Re\left(e^{i\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)}\right) = 4\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right) \Rightarrow \begin{cases} \alpha = 4 \\ \beta = \frac{\pi}{2} \\ \gamma = \frac{\pi}{4} \end{cases}$$