Math 104-1
Midterm 2
October, 2015

Name: ____

- You will have **50 minutes** to complete the exam. The start time and end time will be signaled by the instructor. Do not open the exam or write anything on the exam, including on this cover sheet, until the exam has begun.
- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. Unless stated otherwise, you may use any result proved in class, the text, or in homeworks, but be sure to clearly state the result before using it and to verify that all hypotheses are satisfied.
- This is a closed-book, closed notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- Unless specified otherwise, you may assume the derivatives and/or integrals of common functions like $x^n, e^x, \cos x, \sin x, \tan x$ and their inverses.
- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.
- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

After reading the above instructions, please sign the following:

On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ____

1. Determine whether the following statements are true or false. No justification is required.

   (a) (2 points) Every bounded sequence has a convergent subsequence.

   TRUE false

   (b) (2 points) The series $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}\right)$ can be rearranged to converge to $e^{\pi^{100}} - \sqrt{2}$.

   TRUE false

   (c) (2 points) If f is continuous on $[a, b]$ and $f(a) < c < f(b)$, there exists $x \in (a, b)$ so that $f(x) = c$.

   TRUE false

2. Define the following terms.

   (a) (3 points) continuous function (at x_0)

   Solution: Definition 1: A function $f : X \to \mathbb{R}$ is continuous at x_0 if for every sequence (x_n) converging to x_0, we have that $\lim_{n\to\infty} f(x_n) = f(x_0)$.
   Definition 2: A function $f : X \to \mathbb{R}$ is continuous at x_0 if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $x \in X$ and $|x - x_0| < \delta$ implies $|f(x) - f(x)| < \epsilon$.

   (b) (3 points) uniform continuity

   Solution: A function f is uniformly continuous on S if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $x, y \in S$ and $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$.

   (c) (3 points) strictly increasing function

   Solution: A function f is strictly increasing if $x < y$ implies that $f(x) < f(y)$.

3. (10 points) Prove the following version of the comparison test, without referring to the comparison test: If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and $|b_n| \leq |a_n|$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} b_n$ is absolutely convergent.

Solution: $\sum a_n$ being absolutely convergent means that $\sum |a_n|$ converges. But a series converges if and only if it is a Cauchy series (Ross 14.4), so given $\epsilon > 0$, there exists an N such that $n \geq m > N$ implies that $|\sum_{k=m}^{n} |a_k|| < \epsilon$. But since $|a_k| \geq 0$ for all k, we have that

$$|\sum_{k=m}^{n} |a_k|| = \sum_{k=m}^{n} |a_k| < \epsilon.$$

Then, $|b_n| \leq |a_n|$ implies that

$$|\sum_{k=m}^{n} |b_k|| = \sum_{k=m}^{n} |b_k| \leq \sum_{k=m}^{n} |a_k| < \epsilon,$$

so that $\sum |b_k|$ is Cauchy, hence convergent. Thus, $\sum b_k$ is absolutely convergent.

4. (10 points) A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* if there exists a $T > 0$ such that $f(x) = f(x+T)$ for all $x \in \mathbb{R}$. Suppose that f is a periodic function that is differentiable on $\mathbb{R}$. Show that there exists $x \in \mathbb{R}$ such that $f'(x) = 0$.

Solution: Since f is periodic, there exists $T > 0$ such that $f(x+T) = f(x)$ for all x. In particular, $f(0) = f(T)$. Since f is differentiable for every $x \in \mathbb{R}$, f is continuous at every $x \in \mathbb{R}$ by Ross 28.2 – in particular, f is continuous on $[0, T]$. Moreover, f is differentiable on $\mathbb{R}$, so f is differentiable on $(0, T)$.

Thus, we can apply the Mean Value Theorem / Rolle's Theorem (Ross 29.3/29.2) on $[0, T]$, which then states that there exists $x \in (a, b)$ such that

$$f'(x) = \frac{f(T) - f(0)}{T - 0} = 0.$$

5. Let $[a, b]$ be an interval and $c \in (a, b)$. Define a function f on $[a, b]$ as follows:

$$f(x) = \begin{cases} 0, & x \neq c \\ 1, & x = c \end{cases} .$$

(a) (10 points) Show that f is integrable on $[a, b]$, and find $\int_a^b f$.

Solution: Let $0 < \epsilon < \min\{|c-a|, |c-b|\}$ and take P_ϵ to be the partition $\{a = t_0, t_1 = c - \epsilon/2, t_2 = c + \epsilon/2, t_3 = b\}$.
Then, for $x \in [t_0, t_1] = [a, c-\epsilon/2]$, we have that $f(x) = 0$, so that $M(f, [t_0, t_1]) = m(f, [t_0, t_1]) = 0$. Similarly, for all $x \in [t2, t_3]$, we have that $f(x) = 0$, so that $M(f, [t_2, t_3]) = m(f, [t_2, t_3]) = 0$.
On the interval $[t_1, t_2]$, we have that $f(x) = 1$ if $x = c$ and $f(x) = 0$ otherwise, so that $M(f, [t_1, t_2]) = 1$ and $m(f, [t_1, t_2]) = 0$.
Hence,

$$\begin{aligned} U(f, P_\epsilon) &= M(f, [t_0, t_1])(t_1 - t_0) + M(f, [t_1, t_2])(t_2 - t_1) + M(f, [t_2, t_3])(t_3 - t_2) \\ &= 0 + 1(t_2 - t_1) + 0 \\ &= 0 + 1[c + \epsilon/2 - (c - \epsilon/2)] + 0 = \epsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} L(f, P_\epsilon) &= m(f, [t_0, t_1])(t_1 - t_0) + m(f, [t_1, t_2])(t_2 - t_1) + m(f, [t_2, t_3])(t_3 - t_2) \\ &= 0 + 0 + 0 = 0. \end{aligned}$$

Since $U(f) = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}$, for every $\epsilon > 0$, we have that $U(f) \leq U(f, P_\epsilon) = \epsilon$. Thus, $U(f) \leq 0$ (see Exercise 3.8 from Homework 2). Similarly, as $L(f) = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}$, we have that $L(f) \geq L(f, P_\epsilon) = 0$, so that $L(f) \geq 0 \geq U(f)$. Since we have from Ross 32.8 that $U(f) \geq L(f)$ (noting that f is bounded), then $U(f) = L(f)$, so that f is integrable on $[a, b]$.
Furthermore, we must have that $\int_a^b f = U(f) = L(f) = 0$.

(b) (5 points) Let g be an integrable function on $[a, b]$ and suppose that $h(x) = g(x)$ for all but one x in $[a, b]$. Show that h is integrable and that $\int_a^b g = \int_a^b h$.

Solution: Take $k(x) = h(x) - g(x)$. By assumption, we have that $k(x) = 0$ for all but one $x \in [a, b]$, say $x = c$. Then, if we let $h(c) - g(c) = C$, we have that $k(x) = C$ if $x = c$ and $k(x) = 0$ otherwise.

Thus, $k(x) = Cf(x)$, where $f(x)$ is defined as in part (a). We have that $f(x)$ is integrable, and any constant multiple of an integrable function is integrable, so $Cf(x)$ is integrable wit $\int_a^b Cf = C\int_a^b f = 0$.

We also have that $h(x) = g(x) + k(x)$ is the sum of integrable functions, so is integrable By Ross 33.3, and $\int_a^b h = \int_a^b g + \int_a^b k = \int_a^b g + 0 = \int_a^b g$, as desired.

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Question:	1	2	3	4	5	Total
Points:	6	9	10	10	15	50
Score:						