PHYSICS 110A MIDTERM

Prof. C. F. McKee — Oct. 20, 2006

Partial credit will be given, so show your reasoning carefully. The number of points for each problem is listed at the left. You are permitted 1 sheet of notes, written on two sides.

1. Vector math

(5) a) Evaluate $\hat{\boldsymbol{r}}$ at $\boldsymbol{r} = (1, 2, 2)$

(10) b) Give the general solution for $r > 1$ for the function $\phi(\boldsymbol{r})$ that satisfies

$$\nabla^2\phi = 2, \quad r \leq 1, \quad \nabla^2\phi = 0, \quad r > 1. \tag{1}$$

In other words, you do not need to give the solution for $r \leq 1$. You will receive partial credit if you give your answer in terms of a volume integral over a specified volume.

Solution: a)

$$\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r} \tag{2}$$

$$r^2 = 1 + 4 + 4 = 9 \quad \Rightarrow \quad r = 3 \tag{3}$$

$$\hat{\boldsymbol{r}} = \frac{1}{3}(1, 2, 2). \tag{4}$$

b) The formal solution is

$$\phi(\boldsymbol{r}) = -\frac{1}{4\pi}\int_{r'<1} \frac{2}{r_{\rm sf}}d\tau', \tag{5}$$

where $r_{\rm sf} \equiv |\boldsymbol{r} - \boldsymbol{r}'|$. This is equivalent to a charge distribution with $\rho/\epsilon_0 = -2$ for $r \leq 1 \Rightarrow$ the total charge is $Q = 4\pi\rho/3 = -8\pi\epsilon_0/3$. Thus, the solution for $r > 1$ is

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} = -\frac{2}{3r}. \tag{6}$$

2. Electrostatic potential

(20) Find the potential inside a uniformly charged sphere (i.e., the charge fills the volume of the sphere) with a radius R and a total charge Q. Set the potential at infinity equal to 0.

Solution: For $r \geq R$,

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\boldsymbol{r}} \quad \Rightarrow \quad V = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}. \tag{7}$$

For $r < R$, the charge inside r is $q(r) = (r/R)^3 Q$, so that

$$\begin{aligned}
\boldsymbol{E} &= \frac{1}{4\pi\epsilon_0}\frac{q(r)}{r^2}\,\hat{\boldsymbol{r}} = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}\,\hat{\boldsymbol{r}} \\
\Rightarrow V(r) &= V(R) - \frac{1}{4\pi\epsilon_0}\int_R^r \frac{Qr}{R^3}\,dr \\
&= \frac{1}{4\pi\epsilon_0}\frac{Q}{R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right).
\end{aligned}$$

3. Force between an atom and an ion

(25) Find the force between an ion of charge q and an atom of polarizability α that are separated by a distance r. Is the force attractive or repulsive?

Solution: The ion induces a dipole $\boldsymbol{p} = \alpha \boldsymbol{E}$ in the atom. The force that the ion exerts on the dipole is

$$\boldsymbol{F} = (\boldsymbol{p}\cdot\nabla)\boldsymbol{E} = \alpha(\boldsymbol{E}\cdot\nabla)\boldsymbol{E}, \tag{8}$$

where

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\boldsymbol{r}}. \tag{9}$$

Note that since $\boldsymbol{p} \propto \boldsymbol{E}$, it is not constant, so one cannot use the expression $\boldsymbol{F} = \nabla(\boldsymbol{p}\cdot\boldsymbol{E})$. Set up a Cartesion coordinate system with $\hat{\boldsymbol{x}}$ in the direction of the atom from the ion; then

$$\begin{aligned}
\boldsymbol{F} &= \alpha\left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{q}{x^2}\frac{\partial}{\partial x}\frac{q}{x^2} && (10) \\
&= -2\alpha\left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{1}{r^5}, && (11)
\end{aligned}$$

where we replaced x by the separation r in the final step.

4. Hall effect

A current I flows to the right through a rectangular bar of conducting material in the presence of a uniform magnetic field B_0 perpendicular to the current (see figure). The magnetic field deflects the charge, leading to an accumulation of charge on two opposite sides of the bar. This charge produces an electric field that exactly counteracts the magnetic force, so that the current can flow parallel to the bar. Assume that the charge carriers are electrons, with a density n and charge $-e$.

(5) a) Which surfaces do the charges accumulate on?

(10) b) What is the potential difference across the bar (the Hall voltage)?

(5) c) What is the magnitude of the surface charge?

(20) d) The current in the bar generates a magnetic field of its own, which we have neglected. For a current of 1 A flowing through a bar of width $w = 1$ m, what is the magnitude and direction of this field above and below the bar? (You do not need to calculate the field inside the bar.) Is this field indeed negligible if $B_0 = 1$ T?

Solution: a) The charges accumulate on the top and bottom since $\boldsymbol{F} = q\boldsymbol{v}\times\boldsymbol{B}$ is vertical.

b) The electric field builds up until

$$\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} = 0. \tag{12}$$

Set up a coordinate system so that I is in the y-direction and $\boldsymbol{B}$ is in the x-direction; then

$$\boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B} = vB\hat{\boldsymbol{z}}. \tag{13}$$

Note that since the current is carried by electrons, a positive current corresponds to a negative electron velocity. The top of the bar is at a potential

$$V = -\int_{\text{bottom}}^{\text{top}} E dz = -Eh = -vBh \tag{14}$$

relative to the bottom. Since $I = Ja = -neva = -nevwh$, where $a = wh$ is the cross-sectional area of the bar, it follows that

$$V = \frac{IB}{new}, \tag{15}$$

which is positive. This is the Hall voltage. Note that if the charge carriers were positive, e would have the opposite sign and the top would be at a lower voltage than the bottom.

c) Gauss' law implies the boundary condition

$$E_{\perp 1} - E_{\perp 2} = \frac{\sigma}{\epsilon_0}. \tag{16}$$

Apply this to the top surface of the bar, so that E_2 is the field in the bar. The field is perpendicular to this surface, so that $E_\perp = E$. The bar is like a capacitor with no net charge; it follows that $E_1 = 0$ and

$$E_2 = -\frac{\sigma}{\epsilon_0} \quad\Rightarrow\quad \sigma = -\epsilon_0 E, \tag{17}$$

where σ is the surface charge on the top surface. We just found that $E = vB = -IB/newh$, so that

$$\sigma = \frac{\epsilon_0 IB}{newh}, \tag{18}$$

which is positive, as expected from the answer to part b.

d) Label the region above the bar "1" and that below the bar "2". The jump condition for the field is

$$\boldsymbol{B}_{\parallel 1} - \boldsymbol{B}_{\parallel 2} = \mu_0 \boldsymbol{K}\times\hat{\boldsymbol{n}}_1. \tag{19}$$

By symmetry, $|B_1| = |B_2|$; furthermore, the right-hand rule implies that the field above the bar is directed in the opposite direction to that below the bar, $B_1 = -B_2$. It follows that

$$\boldsymbol{B}_1 = \frac{\mu_0 K}{2}\hat{\boldsymbol{x}}. \tag{20}$$

Now, the surface current is

$$K = \frac{I}{w} = 1 \text{ A m}^{-1}, \tag{21}$$

so that

$$B_1 = \frac{1}{2} \times 4\pi \times 10^{-7} \quad \text{T}, \tag{22}$$

which is far smaller than the 1 T field in the bar.

Constants:

$\frac{1}{4\pi\epsilon_0} = 9 \times 10^9$ N m^2/C^2

$\mu_0 = 4\pi \times 10^{-7}$ N/A^2