Not Your CS 70

PROFESSOR KENNETH A. RIBET

Last Midterm Examination

April 2, 2015

2:10–3:30 PM, 100 Lewis Hall

Please put away all books, calculators, cell phones and other devices. You may consult a single two-sided sheet of notes. Please write carefully and clearly in *complete sentences*; this is even more important than it was on the first midterm. Explain what you are doing: the paper you hand in will be your only representative when your work is graded. Do not worry about simplifying or evaluating expressions with decimal numbers, factorials, binomial coefficients and the like. At the conclusion of the exam, hand your paper in to your GSI.

There were five problems, each worth six points.

1. Eve has a bag of 12 biased coins. Six of these come up heads 3/5 of the time, while the other six are come up tails 2/3 of the time. Eve reaches into the bag, pulls out a coin at random and tosses it. The coin comes up heads. What is the probability that she pulled out a coin that is biased toward heads?

This is your classic Bayes Rule problem. Let E be the event that Eve pulled out a head-biased coin; let F be the event that the coin she tossed yielded a head. We have to compute $P(E|F)$, which we can do if we know $P(F|E)$, $P(E)$ and $P(F)$.

We are given that $P(F|E) = 3/5$. Also $P(E) = 1/2$ because it's equally likely that she pulled out a head-leaning coin as a tail-leaning coin. The only hard term is $P(F)$, which we know to compute as $P(F \cap E) + P(F \cap \overline{E})$; here, $\overline{E}$ is the event that she pulled out a tail-biased coin.

Now

$$P(F \cap E) = P(F|E)P(E) = 3/5 \cdot 1/2 = \frac{3}{10}$$

and

$$P(F \cap \overline{E}) = P(F|\overline{E})P(\overline{E}) = 1/3 \cdot 1/2 = 1/6.$$

Thus $P(F) = \dfrac{1}{6} + \dfrac{3}{10} = \dfrac{7}{15}$.

Finally,

$$P(E|F) = P(F|E)\frac{P(E)}{P(F)} = \frac{3}{5} \cdot \frac{1/2}{7/15} = \frac{9}{14},$$

if I didn't make a mistake here.

2a. Alice's five-card poker hand contains the ace of spades and the ace of diamonds. Find the probability that it contains all four aces.

The number of poker hands containing all four aces is 48: you just have to choose the final, "fifth," card to go with the four aces.

The number of poker hands containing the two given aces is $\binom{50}{3} = 19600$ because you need to pick three additional cards, and there are $52 - 2 = 50$ possibilities.

The desired probability is then $48/19600 = \dfrac{3}{1225}$.

b. Bob's five-card poker hand contains at least two aces. Find the probability that it contains all four aces.

Again, there are 48 hands with all four aces. How many hands have two or more aces? The number of hands with exactly two aces is $\binom{4}{2}\binom{48}{3}$. The number of hands with exactly three aces is $\binom{4}{3}\binom{48}{2}$. Hence the number of hands with two or more aces is $\binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + 48 = 108336$. The desired probability is $\dfrac{48}{108336} = \dfrac{1}{2257}$.

3. For each $n \geq 0$, let c_n be the number of ways of writing n as a sum of 1's and 2's; order counts. Thus $c_0 = c_1 = 1$. Also, $c_2 = 2$ because we can write 2 as $1+1$ or simply as 2. We have $c_3 = 3$ because we can write 3 as $1+1+1$, as $1+2$ and as $2+1$. Similarly, 4 may be written

$$1+1+1+1, \quad 2+1+1, \quad 1+2+1, \quad 1+1+2, \quad 2+2;$$

accordingly, $c_4 = 5$.

Use strong induction to prove that c_n is the $(n+1)$st Fibonacci number f_{n+1} for all $n \geq 0$.

The cases $n = 0, 1, 2, 3, 4$ that are discussed in the statement of the problem will serve well as base cases for our induction! From the perspective of strong induction, it is enough to prove, for $n \geq 0$, that the equation $c_{n+2} = f_{n+3}$ is a consequence of the two preceding equations $c_n = f_{n+1}$ and $c_{n+1} = f_{n+2}$.

For this, we note the key Fibonacci-like identity $c_{n+2} = c_{n+1} + c_n$. This identity is seen as follows: when we write $n+2$ as a sum of 1's and 2's, we can begin with a 1 or with a 2. If we begin with a 1, then we have

$$n + 2 = 1 + \text{a sum of 1's and 2's that adds to } n + 1.$$

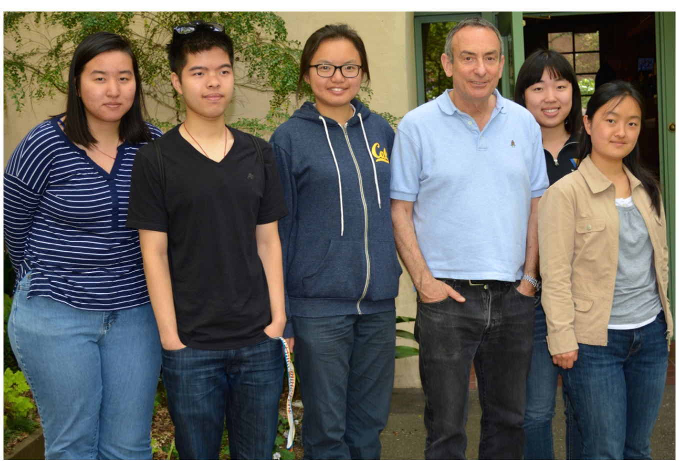
If we begin with a 2, we are writing

$$n + 2 = 2 + \text{a sum of 1's and 2's that adds to } n.$$

The number of ways to express $n+2$ as a sum in the first manner is the number of ways of writing $n+1$ as a sum of 1's and 2's, namely c_{n+1}. Similarly, the number of ways of writing $n+2$ in the second manner is c_n. Since c_{n+2} is the total number of ways of expressing $n+2$ in 1's and 2's, we have indeed $c_{n+2} = c_{n+1} + c + n$.

Assuming $c_n = f_{n+1}$ and $c_{n+1} = f_{n+2}$, we can then write $c_{n+2} = c_{n+1} + c_n = f_{n+2} + f_{n+1} = f_{n+3}$, where the final equality comes from the recursive formula defining the Fibonacci sequence. Comparing the first and last terms of the chain, we see that we have $c_{n+2} = f_{n+3}$, which was the formula to be obtained.

By the way, I discussed this problem in class one day when presenting the Fibonacci numbers.



4a. In how many ways can a class of 240 students be divided into 10 equally-sized discussion sections?

This is a problem of the sort that has been discussed repreatedly on `piazza` over the last week or two. One way to do it is to put students into "numbered" groups 1–10 and then to divide by 10! at the end to take into account that the discussion sections were intended to be indistinguishable. Whoa, wait a minute: how do we know that the discussion sections were intended to be indistinguishable? That's a good question! OK, never mind: I'll take off 0 points if you think that they are distinguishable.

When the groups are numbered, there are $\binom{240}{24}$ ways to fill up the first discussion group, $\binom{240-24}{24}$ ways to then fill up the second group, $\binom{240-48}{24}$ ways to fill up the third, and so on. The answer may thus be written $\prod_{i=0}^{9}\binom{240-24i}{24}$. The final answer was intended to be $\frac{1}{10!}\prod_{i=0}^{9}\binom{240-24i}{24}$, but you don't really have to divide by 10!.

b. In how many ways can 240 identical chairs be placed in 10 (capacious) classrooms so that each classroom gets at least 20 chairs?

This is a bagel problem. You need to introduce nine dividers to separate chairs into ten classrooms. The way to think about this problem is to first place 20 chairs in each of the 10 classrooms. This leaves $240 - 20 \cdot 10 = 40$ chairs to be distributed among the classrooms. Thus the answer is $\binom{40+9}{9}$.

5. Let $p \geq 7$ be a prime number. Show that the $(p{-}1)$-digit number $11\cdots 11$ is a multiple of p. For example the six-digit number 111111 is divisible by 7, the 10-digit number 1111111111 is divisible by 11, and the 12-digit number 111111111111 is divisible by 13.

According to http://en.wikipedia.org/wiki/Repunit, a number of the form $11\cdots 1$ is called a *repunit*. I'm sure that you wanted to know that.

Since $p \neq 2, 5$, $\gcd(p, 10) = 1$, and thus $10^{p-1} \equiv 1 \pmod{p}$ by Fermat's Little Theorem. The number $10^{p-1} - 1$ has the decimal representation $99\cdots 9$ with $p{-}1$ 9's. Thus p divides $99\cdots 9 = 9 \times 11\cdot 1$ (with $p{-}1$ 1's). Since $p \neq 3$, $\gcd(p, 9) = 1$, and thus p divides $11\cdots 1$ by Euclid's Lemma.

One of the (U)GSIs claimed that this problem would be hard.