University of California, Berkeley
Department of Mechanical Engineering
ME185, Fall 2015

Midterm Exam (22 Oct 2015)

Consider a deformable continuum $\mathscr{B}$ undergoing a motion $\boldsymbol{\chi}(\mathbf{X}, t)$, $t \geq 0$, with

$$\boldsymbol{\chi}(\mathbf{X}, 0) = \mathbf{X}. \tag{1}$$

The velocity field is

$$\mathbf{v} = \frac{\partial \boldsymbol{\chi}}{\partial t}(\mathbf{X}, t). \tag{2}$$

Consider a differential element $d\mathbf{X}$ (i.e., a tangent vector to a fixed curve in the reference configuration of $\mathscr{B}$) at the location $\mathbf{X}$:

$$d\mathbf{X} = \mathbf{M}\, dS \quad (\mathbf{M} \cdot \mathbf{M} = 1), \qquad dX_A = M_A dS. \tag{3}$$

This element is mapped linearly by the deformation gradient

$$\mathbf{F} = \frac{\partial x_i}{\partial X_A} \mathbf{e}_i \otimes \mathbf{E}_A, \qquad (J = \det \mathbf{F} > 0) \tag{4}$$

into the differential element

$$d\mathbf{x} = \mathbf{m}\, ds \quad (\mathbf{m} \cdot \mathbf{m} = 1), \qquad dx_i = m_i ds, \tag{5}$$

at the location $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ at time t. Recall the polar decomposition theorem

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}. \tag{6}$$

<u>**Problem 1**</u> (25 points)

(a) Show that

$$\lambda \mathbf{m} = \mathbf{F}\mathbf{M}, \quad \text{or} \quad \lambda m_i = F_{iA} M_A, \tag{7}$$

where λ is the stretch function.

(b) Deduce that

$$\lambda^2 = \mathbf{M} \cdot \mathbf{C}\mathbf{M} = C_{AB} M_A M_B \tag{8}$$

where

$$\mathbf{C} = \mathbf{F}^T \mathbf{F}, \qquad C_{AB} = F_{iA} F_{iB} \tag{9}$$

is the right Cauchy-Green tensor.

(c) Consider the motion described on a fixed orthonormal basis $\mathbf{e}_i = \mathbf{E}_i$ by

$$\begin{aligned} x_1 &= (1+t)X_1 + 2tX_2, \\ x_2 &= tX_1 + X_2, \\ x_3 &= (1+3t)X_3, \end{aligned} \tag{10}$$

Calculate **F**, J, and **C**.

(d) Calculate the velocity field in Lagrangian form.

(e) For the motion (10), calculate the stretch at $t = \frac{1}{2}$ of the material element that lies along the direction $\mathbf{E}_1$ in the reference configuration of $\mathscr{B}$.

Problem 2 (20 points)

Suppose that the components of the velocity field for the body $\mathscr{B}$ is given in Eulerian form by

$$\begin{aligned} v_1 &= ctx_1^2 x_2^2, \\ v_2 &= -\frac{1}{3} t x_1 x_2^3, \\ v_3 &= 0. \end{aligned} \tag{11}$$

(a) Calculate the components of the acceleration field.

(b) If the motion is isochoric, solve for the constant c.

(c) For the velocity field in Part (b), calculate the vorticity vector

$$\boldsymbol{\omega} = \frac{1}{2} \operatorname{curl} \mathbf{v}, \quad \text{or} \quad \omega_i = \frac{1}{2} e_{ijk} v_{k,j}. \tag{12}$$

(d) Check that for the vorticity field in Part (c), $\operatorname{div} \boldsymbol{\omega} = 0$.

Problem 3 (15 points)

Recall that for a scalar-valued function $\phi = \tilde{\phi}(\mathbf{x}, t)$, at any time t, the directional derivative of ϕ at $\mathbf{x}_0$ in the direction of an arbitrary non-zero vector $\mathbf{h}$ is given by

$$\delta\phi(\mathbf{x}_0, \mathbf{h}, t) = \frac{d}{d\xi} \phi\left(\mathbf{x}_0 + \xi \mathbf{h}, t\right)\Big|_{\xi=0} = \nabla\phi \cdot \mathbf{h}. \tag{13}$$

(a) For the function

$$\phi = \tilde{\phi}(\mathbf{x}, t) = ct\,\mathbf{x} \cdot \mathbf{x} = ctx_i x_i, \tag{14}$$

where c is a constant, calculate the value of $\nabla\phi$ at the position $\mathbf{x}_0 = \mathbf{e}_1 + 3\mathbf{e}_2$.

(b) Hence, evaluate $\delta\phi\left(\mathbf{x}_0, \mathbf{e}_2, t\right)$.

(c) For a tensor field

$$\mathbf{A} = x_1^2 x_2\, \mathbf{e}_1 \otimes \mathbf{e}_1 + 2x_1 x_2^2\, \mathbf{e}_1 \otimes \mathbf{e}_2 + x_2 x_3^2\, \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{15}$$

calculate

$$\operatorname{div} \mathbf{A} = A_{ij,j}\mathbf{e}_i, \tag{16}$$

and evaluate it at the position $\mathbf{x} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$.

Problem 4 (25 points)

Suppose that the Cauchy-Green tensor $\mathbf{C}$ and the rotation tensor $\mathbf{R}$ at $(\mathbf{X}, t)$ are given by

$$\mathbf{C} = 4(\mathbf{E}_1 \otimes \mathbf{E}_1 + \mathbf{E}_2 \otimes \mathbf{E}_2) + \mu^2\,\mathbf{E}_3 \otimes \mathbf{E}_3, \tag{17}$$

$$\mathbf{R} = \cos\omega t(\mathbf{E}_1 \otimes \mathbf{E}_1 + \mathbf{E}_2 \otimes \mathbf{E}_2) - \sin\omega t(\mathbf{E}_1 \otimes \mathbf{E}_2 - \mathbf{E}_2 \otimes \mathbf{E}_1) + \mathbf{E}_3 \otimes \mathbf{E}_3, \tag{18}$$

where μ and ω are positive constants.

(a) Find the right stretch tensor $\mathbf{U}$.

(b) What are the eigenvalues and eigenvectors of $\mathbf{U}$?

(c) Calculate the deformation gradient $\mathbf{F}$.

(d) If the motion is isochoric, solve for μ.

(e) Calculate the left stretch tensor $\mathbf{V}$.

(f) Identify an eigenvalue and eigenvector of $\mathbf{F}$.