DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

Name and section: ____

GSI's name: ____

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This exam consists of 5 questions. Answer the questions in the spaces provided.

1. (a) (15 points) Find a general solution to the following differential equation:

$$y' + y + e^t = 0.$$

Solution:

$y' + y + e^t = 0 \Rightarrow y' + y = -e^t$

Let $a(t) = 1$, $b(t) = -e^t$, $A(t) = t$

$\Rightarrow y = \frac{1}{e^t}\int e^t \cdot (-e^t)\,dt = \frac{1}{e^t}\int -e^{2t}\,dt$

$= \frac{1}{e^t}\left(\frac{-1}{2}e^{2t} + c\right) = \frac{-e^t}{2} + \frac{c}{e^t}$ //

(b) (5 points) Using part(a) find a solution which satisfies the initial condition

$$y(0) = -1.$$

Solution:

$y(0) = \frac{-e^0}{2} + \frac{c}{e^0} = \frac{-1}{2} + \frac{c}{1}$

$y(0) = -1 \Rightarrow \frac{-1}{2} + c = -1 \Rightarrow c = \frac{-1}{2}$

$\Rightarrow y = \frac{-e^t}{2} - \frac{1}{2e^t}$ //

2. Compute the following integrals:

(a) (10 points)

$$\int x\sec^2(x^2)dx$$

Solution:

Let $u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

$\Rightarrow \int x\sec^2(x^2)\,dx = \int \frac{1}{2}\sec^2(u)\,du = \frac{1}{2}\tan(u) + C$

$= \frac{1}{2}\tan(x^2) + C$

(b) (10 points)

$$\int x^2\ln(x)dx.$$

Solution:

Let $f(x) = \ln(x)$, $g(x) = x^2$, $G(x) = \frac{1}{3}x^3$

$f'(x) = \frac{1}{x}$

$\Rightarrow \int x^2\ln(x)\,dx = \ln(x)\cdot\frac{1}{3}x^3 - \int \frac{1}{3}x^2\,dx$

$= \frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3 + C$

3. (20 points) Determine if the following improper integral is convergent or divergent.

$$\int_0^\infty \frac{2}{(x+4)^2}dx.$$

If it is convergent determine its value.

Solution:

$$\int_0^\infty \frac{2}{(x+4)^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{2}{(x+4)^2}\,dx$$

$$= \lim_{t\to\infty} \left(\frac{-2}{(x+4)} \Big|_0^t \right)$$

$$= \lim_{t\to\infty} \left(\frac{1}{2} - \frac{2}{t+4} \right)$$

$\frac{2}{t+4} \to 0$ as $t \to \infty$ hence $\int_0^\infty \frac{2}{(x+4)^2}\,dx = \frac{1}{2}$ and the integral is convergent.

4. (a) (10 points) Find a general solution to the following differential equation:

$$yy' = t^2 + 1$$

Solution:

First observe that if $y = k$, a constant solution $\Rightarrow y' = 0$ $\Rightarrow 0 = t^2 + 1$. This is not true, hence no constant solution.

$y\frac{dy}{dt} = t^2 + 1 \Rightarrow \int y\,dy = \int (t^2+1)\,dt \Rightarrow \frac{1}{2}y^2 = \frac{1}{3}t^3 + t + C$

$\Rightarrow y = \pm\sqrt{\left(\frac{2}{3}t^3 + 2t + 2C\right)}$ //

(b) (5 points) Using part(a) find a solution which satisfies the initial condition

$$y(0) = -2.$$

Solution:

$y(0) = \pm\sqrt{2C}$. Thus $y(0) = -2 \Rightarrow$

$-\sqrt{2C} = -2 \Rightarrow 2C = 4 \Rightarrow y = -\sqrt{\frac{2}{3}t^3 + 2t + 4}$ //

(c) (5 points) Using part(a) find a solution which satisfies the initial condition

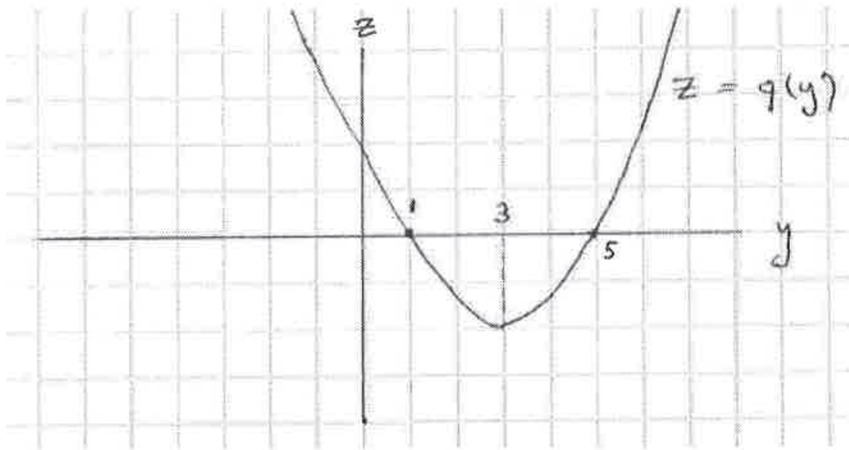
$$y(1) = 0.$$

Solution:

$y(1) = \pm\sqrt{\frac{2}{3} + 2 + 2C}$. Hence $y(1) = 0 \Rightarrow \frac{2}{3} + 2 + 2C = 0$

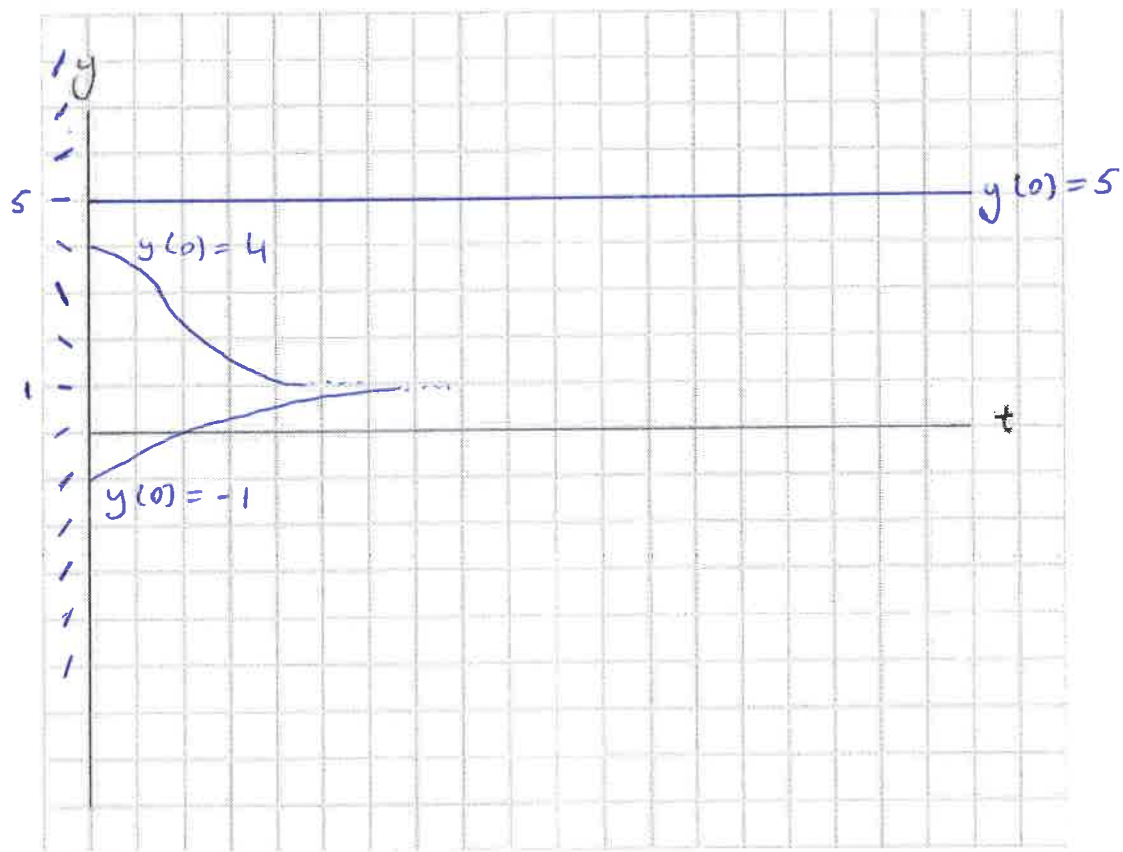
$\Rightarrow 2C = \frac{-8}{3} \Rightarrow y = \pm\sqrt{\frac{2}{3}t^3 + 2t - \frac{8}{3}}$ //

5. (20 points) Consider the differential equation of the form $y' = q(y)$, where the graph $z = q(y)$ is as follows:



Sketch a solution for each of the following initial conditions: $y(0) = -2$, $y(0) = 5$ and $y(0) = 4$.

Solution:



END OF EXAM

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