Final Exam

Physics 105, Fall 04; Instructor: Petr Hořava
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Problem 1. Consider the following time-dependent Lagrangian for a system with one degree of freedom q,

$$\mathcal{L} = e^{\beta t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right), \tag{1}$$

with β a fixed real constant greater than zero (and of course m and k fixed constants greater than zero).

1(a) Write down the Euler-Lagrange equations of motion for this system, and interpret the resulting equations in terms of known physical systems.

Solution

The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) - \frac{\partial\mathcal{L}}{\partial q} = e^{\beta t}(m\ddot{q} + m\beta\dot{q} + kq) = 0.$$

The multiplicative factor $e^{\beta t}$ is non-zero for finite t and can be dropped (since it does not influence the validity of the EoM), leading to the equation of motion of a conventional damped harmonic linear oscillator, with β determining the (inverse) "quality" of the oscillator.

1(b) Introduce a new coordinate $Q = e^{\beta t/2}q$, and rewrite the Lagrangian in terms of Q and $\dot{Q}$. Find a continuous symmetry of the Lagrangian in these new variables, derive the corresponding conserved quantity (using the logic of Noether's theorem), and interpret this quantity in terms of the original variable q.

Solution

Substituting Q into the Lagrangian, we get

$$\mathcal{L} = \frac{1}{2}m\left(\dot{Q} - \frac{\beta}{2}Q\right)^2 - \frac{1}{2}kQ^2.$$

We can further simplify this as follows: the term proportional to $\dot{Q}Q$ can and should be dropped (since it is just a total time derivative), leading to

$$\mathcal{L} = \frac{1}{2}m\dot{Q}^2 - \frac{1}{2}\left(k - \frac{m\beta^2}{4}\right)Q^2.$$

This is just the standard simple linear harmonic oscillator! (The only trace of the dissipation constant β is now in the frequency of the Q oscillator. We can easily see how the classification of damped oscillators into underdamped, critically damped, and overdamped nicely translates in the language of the Q variable into whether or not $Q = 0$ is a stable equilibrium point.)

In the Q variables, the Lagrangian is time independent, or in other words, time translations are a continuous symmetry. This implies that the Hamiltonian

$$\mathcal{H} = \frac{1}{2} m \dot{Q}^2 + \frac{1}{2}\left(k - \frac{m\beta^2}{4}\right) Q^2.$$

will be conserved. One can easily rewrite this conserved quantity in terms of q, by simply substituting again $Q = e^{\beta t/2} q$. In the q variables, the conservation law will then control the way the amplitude of the damped oscillator q falls off exponentially with time.

Problem 2. Consider a simple harmonic oscillator, with unit mass and unit frequency, which is undergoing for $t < 0$ simple oscillations described by the trajectory

$$x(t) = x_0 \cos(t) + v_0 \sin(t). \tag{2}$$

(Note that the real constants x_0 and v_0 in Eqn. (2) can be interpreted as the coordinate and the velocity at $t = 0$.) At time $t = 0$, an outside driving force $F(t) = e^{-\gamma t}$ (with γ a real, positive constant) starts acting on the oscillator.

2(a) Using the method of Green's functions, find the trajectory $x(t)$ of the system for $t > 0$.

Solution

The Green's function that we will use is

$$G(t - t') = \sin(t - t').$$

The solution of our problem will then be

$$x(t) = x_0 \cos(t) + v_0 \sin(t) + \int_0^t e^{-\gamma t'} \sin(t - t')\, dt'.$$

(Here the third term solves the equations of motion with the initial condition $x(0) = 0$, $\dot{x}(0) = 0$. In order to match to our intial conditions, we need to add the first two terms, i.e., the solution of the free oscillator problem that respects our required initial value conditions $x(0) = x_0$, $\dot{x}(0) = v_0$.)

To perform the remaining (extremely easy) integral, one can use the "complexifying" trick:

$$\int_0^t e^{-\gamma t'} \sin(t - t')\, dt' = \Im\left\{\int_0^t e^{i(t-t')-\gamma t'}\, dt'\right\}$$

where $\Im$ denotes the imaginary part of what follows. Then

$$\begin{aligned}\int_0^t e^{-\gamma t'} \sin(t - t')\, dt' &= \Im\left\{e^{it}\int_0^t e^{-(i+\gamma)t'}\, dt'\right\} = \Im\left\{e^{it}\frac{1}{i+\gamma}\, e^{-(i+\gamma)t'}\Big|_t^0\right\} \\ &= \Im\left\{e^{it}\frac{\gamma - i}{1+\gamma^2}\left(1 - e^{-(i+\gamma)t}\right)\right\} = \frac{1}{1+\gamma^2}\Im\left\{(\gamma - i)\left(e^{it} - e^{-\gamma t}\right)\right\} = \\ &= \frac{1}{1+\gamma^2}\left(\gamma \sin(t) - \cos(t) + e^{-\gamma t}\right).\end{aligned}$$

2(b) Determine the intitial coordinate x_0 and velocity v_0 at $t = 0$, for which the system will asymptotically reach the state of static equilibrium (i.e., $x(t) = 0$) as $t \to \infty$.

Solution

Using the explicit result of 2(a), we now have our full solution:

$$x(t) = \left(v_0 + \frac{\gamma}{1+\gamma^2}\right)\sin(t) + \left(x_0 - \frac{1}{1+\gamma^2}\right)\cos(t) + \frac{1}{1+\gamma^2}e^{-\gamma t}.$$

The first two of these three terms are oscillatory at all t, while the last one decays exponentially with time. We see that in order for the system to asymptotically approach the point of static equilibrium at $x = 0$ as $t \to \infty$, we must set the coefficients of the first two (oscillatory) terms to zero, leading to

$$x_0 = \frac{1}{1+\gamma^2}, \qquad v_0 = -\frac{\gamma}{1+\gamma^2}.$$

Problem 3. Consider a particle of mass m in three spatial Euclidean dimensions, in cylindrical coordinates r, θ, z (these are related to the conventional Cartesian coordinates x, y, z via $x = r\cos\theta$ and $y = r\sin\theta$). The Lagrangian has the standard form $\mathcal{L} = T - V$, where T is the standard kinetic energy, and the potential V is only a function of r and $k\theta + z$,

$$V = V(r, k\theta + z), \tag{3}$$

where k is some real constant. (For example, one could consider a potential of the form $V_{\text{example1}} = \frac{1}{2}r^2 + k\theta + z$, or perhaps $V_{\text{example2}} = -\frac{M}{r} + \sin(\theta + z/k)$ – the exact form of V doesn't matter.) Find a symmetry of this Lagrangian, and use Noether's theorem to obtain the constant of the motion associated with it.

Solution

Since the potential only depends on θ and z through their combination $k\theta + z$, it will be unchanged under the following symmetry transformation:

$$\theta \to \theta + \alpha, \qquad z \to z - k\alpha,$$

with α an arbitrary real constant. (Indeed, this transformation was designed to keep $k\theta + z$ invariant, which is all we need to prove that V will also be invariant under this tranformation.) This symmetry tranformation can be interpreted as a rotation by an arbitrary angle α around the z axis, *accompanied by a translation* of z by $-k\alpha$ (i.e., a sort of "helical" symmetry).

From Noether's theorem, one can then easily get the conservation law. It is the following combination of the momenta,

$$P \equiv p_\theta - kp_z \equiv mr^2\dot{\theta} - mk\dot{z}$$

that is conserved.

An alternative way of solving the same problem would be to switch to coordinates $\tilde{\theta} = \theta + z/k$, $\tilde{z} = z$, and interpret the symmetry as that of translations along $\tilde{z}$. Getting the conservation law in those coordinates (i.e., the linear momentum $p_{\tilde{z}}$ is conserved) and transforming back to the original variables then reproduces the same conserved quantity P that we found above.

Problem 4. Consider a particle of mass m and electric charge e, moving on a plane x, y, in a constant magnetic field B. We shall choose the vector potential of the electromagnetic field in a gauge such that $A_x = 0$ and $A_y = Bx$; in this gauge, the Lagrangian of the system can be written as

$$\mathcal{L} = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) + eBx\dot{y}. \tag{4}$$

4(a) Derive the canonical momenta and the Hamiltonian for this system.

Solution

We have

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x},$$
$$p_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y} + eBx.$$

The Hamiltonian is then

$$\mathcal{H} = \partial_x \dot{x} + p_y \dot{y} - \mathcal{L} = \frac{p_x^2}{2m} + \frac{(p_y - eBx)^2}{2m}.$$

(It is important to write the Hamiltonian here in the correct variables, i.e., the phase-space momenta and coordinates.)

4(b) Write down the Hamilton-Jacobi equation for this problem, solve it by the separation of variables (reduce the problem at least to quadratures). Interpret physically the conservation laws that emerged in the process of separating the variables.

Solution

The Hamilton-Jacobi equation is an equation for S as a function of t, x and y:

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left\{ \left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} - eBx \right)^2 \right\} = 0.$$

This can be solved by separation of variables, as follows. Assume that S can be written as a sum of three functions, each of which depends only on one of our three variables

t, x, y. Observe further that the equation does not depend explicitly on t or y; thus, the dependence of S on y and t can be at most linear, and we can write

$$S(t, x, y) = W(x) + \alpha_y y - Et,$$

where α_y is a constant. Plugging this back to the HJ equation gives

$$\frac{1}{2m}\left\{\left(\frac{dW}{dx}\right)^2 + (\alpha_y - eBx)^2\right\} = E.$$

This can be rewritten as

$$\frac{dW}{dx} = \sqrt{2mE - (\alpha_y - eBx)^2}.$$

Thus, the problem has been reduced to quadratures. The conservation laws encountered in the process: E corresponds to the conservation of the Hamiltonian, and α_y to the conserved quantity associated with the fact that y is an ignorable coordinate in the original Lagrangian.

4(c) BONUS: For five extra bonus points, solve the equations resulting from 4(b) completely, and use the solution to determine the trajectories of the system.

Solution

In order to determine the profile of the trajectory (for example, y as a function of x), the solution of the Hamilton-Jacobi equation is sufficient in the form presented above, and in fact, it is not even necessary to perform the remaining integral (which I am sure everybody could do, anyway). Indeed, we have a solution S as a function of two arbitrary constants, E and α, with $W(x)$ a solution of

$$\frac{dW}{dx} = \sqrt{2mE - (\alpha_y - eBx)^2}.$$

The trajectory will be determined as a solution of

$$\frac{\partial S(t, x, y; \alpha_y, E)}{\partial \alpha_y} = \beta_y,$$

where we have introduced a new constant β_y. In more detail, the left-hand-side of this equation is

$$\frac{\partial S(t, x, y; \alpha_y, E)}{\partial \alpha_y} = y + \frac{\partial W(x; \alpha_y, E)}{\partial \alpha_y}.$$

Notice that since the right-hand-side of the equation for dW/dx depends only on $\alpha_y - eBx$ and not on α_y and x separately, we have (up to a possible irrelevant additive constant)

$$\frac{dW}{d\alpha_y} = -\frac{1}{eB}\frac{dW}{dx}.$$

This is a fortunate fact – it implies that we don't have to integrate for $W(x)$, because we would then immediately just take the derivative of it again! Hence, the equation we wish to solve can be rewritten as

$$y - \frac{1}{eB}\frac{\partial W(x;\alpha_y,E)}{\partial x} = \beta_y,$$

or (with the use of the above expression for dW/dx)

$$y - \frac{1}{eB}\sqrt{2mE - (\alpha_y - eBx)^2} = \beta_y.$$

After transferring y to the right hand side and taking the square of both sides, we get our equation for the orbits,

$$2mE - (\alpha_y - eBx)^2 = (eB\beta_y - eBy)^2.$$

Hence, the orbits are circles, centered on an arbitrary location in the plane, and with radii set by the energy.

Problem 5. Using the Lagrangian of the particle in constant magnetic field (Eqn. (4) of Problem 4), write down the Euler-Lagrange equations for the system (but don't solve them). Then show that there is a rotating frame (with some constant angular velocity ω_L, called the "Larmor frequency") in which these equations of motion become those of a simple planar harmonic oscillator.

Solution

This problem is explicitly discussed (in the Hamiltonian formalism) in the body of one of the Chapters of [Hand-Finch], so I will be brief.

The Euler-Lagrange equations of motion are

$$m\ddot{x} - eB\dot{y} = 0, \qquad m\ddot{y} + eB\dot{x} = 0.$$

To show that in a frame rotating with some frequency ω_L, this is equivalent to the problem of a planar harmonic oscillator, one can proceed in several different ways. One approach would be to apply to formula that transforms the time derivatives of various vector quantities from the lab frame to the rotating frame, apply it to *all* terms in the equations of motion, and see for which frequency ω_L of the rotating frame the terms proportional to $\dot{x}$ and $\dot{y}$ in the equations of motion drop out. Alternatively, one can introduce new coordinates $\tilde{x}$, $\tilde{y}$ attached to the rotating frame, and derive the equations of motion in terms of the new variables by brute force. Either way, one gets

$$\omega_L = \frac{eB}{2m}.$$

Notice the crucial factor of 1/2 in this frequency: the Larmor frequency ω_L is *one half* of the cyclotron frequency associated with the motion of the particle in the constant magnetic field.

Problem 6. Two observers (let's call them Anežka and Bořivoj) have been asked to examine a rigid body. They are both using Cartesian coordiates to record their observations, but Anežka's Cartesian frame is not necessarily the same as Bořivoj's. Anežka's measurements lead to the moment of inertia given in her Cartesian frame by the following diagonal matrix,

$$I_A = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

while Bořivoj in his frame observes

$$I_B = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{pmatrix}.$$

Is it possible that they are both looking at the same rigid body? (Before answering this question by a specific calculation, explain in a few words your method of approaching the problem.)

Solution

The first thing to test is whether or not the two moments of inertia are related by a rotation, i.e., whether there is an orthogonal tranformation U such that $I_B = U I_A \tilde{U}$. The way to test it is not to search desperately for U, but to check that the eigenvalues of I_A and I_B are the same. I_A is already diagonalized, so we need to test whether or not the eigenvalues of I_B are 3, 6 and 9.

In principle, we could calculate

$$\det \begin{pmatrix} 5-\lambda & -2 & 0 \\ -2 & 6-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{pmatrix} = (5-\lambda)(6-\lambda)(7-\lambda) - 4(5-\lambda) - 4(7-\lambda),$$

simplify it, and find its three roots. However, we don't even need to work so hard, and risk arithmetic errors. Just plugging in 3, 6 and 9 for λ reveals immediately that they all make the determinant vanish, i.e., they represent the eigenvalues of I_B. Hence, Anežka and Bořivoj could indeed be observing the same rigid body, in two frames related to each other by a simple rotation.

Problem 7. A system with three degrees of freedom u, v, w is described in the linearized approximation by the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) - \frac{1}{2}\left(2u^2 - 4uv + 3v^2 + 4vw + 4w^2\right).$$

Determine the normal modes and the corresponding frequencies of the system near $u = v = w = 0$. Discuss qualitatively the character of the motion (i.e., oscillatory, runaway, or linear) for the individual normal modes.

Solution

The problem is already linearized, and we can immediately read off the equation that determines the characteristic frequencies:

$$\det \begin{pmatrix} 2-\omega^2 & -2 & 0 \\ -2 & 3-\omega^2 & 2 \\ 0 & 2 & 4-\omega^2 \end{pmatrix} \equiv \omega^2(\omega^4 - 9\omega^2 + 18) = 0.$$

Hence, we see that $\omega_1 = 0$ is a root, and we only have to find two roots of the remaining quadratic equation; those are easily found to be

$$\omega_2^2 = 3, \qquad \omega_3^2 = 6.$$

Even before calculating the normal modes associated with these frequencies, we see that the first one corresponds to linear motion, while the other two will be oscillatory.

The normal modes can now be determined by the method of cofactors, from the matrix

$$\begin{pmatrix} 2-\omega_{(i)}^2 & -2 & 0 \\ -2 & 3-\omega_{(i)}^2 & 2 \\ 0 & 2 & 4-\omega_{(i)}^2 \end{pmatrix},$$

where the index (i) parametrizes the three frequencies. The result is

$$\Phi_{(1)} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \qquad \Phi_{(2)} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \qquad \Phi_{(3)} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}.$$

This concludes the solution of the final exam.