Math 54. Solutions to Second Midterm

1. (22 points) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(a). Verify that 3 is an eigenvalue of A, and find a basis for its eigenspace.

We have that 3 is an eigenvalue because

$$\det(A - 3I) = \begin{vmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = -8 + 1 + 1 - (-2) - (-2) - (-2) = 0 \,.$$

The eigenspace is the null space of $A - 3I$, so we row-reduce this matrix:

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \,.$$

The solutions of $(A - 3I)\vec{x} = \vec{0}$ are therefore all vectors with $x_1 = x_2 = x_3$, so a basis for the eigenspace is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(b). Find another eigenvalue of A, and find a basis for its eigenspace. [**Hint:** There's an easy way to answer this part.]

It is easy to see that A is singular, so it has 0 as an eigenvalue. Row reducing it gives

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \,,$$

so the solutions of $A\vec{x} = \vec{0}$ are $x_1 = -x_2 - x_3$ with x_2 and x_3 free; writing this in vector parametrized form gives

$$x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \,,$$

so a basis for the eigenspace is $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

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(c). Is A diagonalizable? Explain.

Yes. It has three linearly independent eigenvectors. Or, the dimensions of the eigenspaces add up to three (Theorem 7b on page 255).

(Or, it is a real symmetric matrix, so it is diagonalizable.)

2. (20 points) Let $W = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ and $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

(a). Find $\widehat{y} = \operatorname{proj}_W \vec{y}$.

Note that the two vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ are orthogonal, so

$$\operatorname{proj}_W \vec{y} = \frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \frac{\vec{y} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 = \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} .$$

(b). Find the smallest value of $\|\vec{y} - \vec{v}\|$ for $\vec{v} \in W$.

That smallest value is when $\vec{v} = \widehat{y}$, so it is

$$\|\vec{y} - \widehat{y}\| = \left\| \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\| = \sqrt{6} .$$

3. (18 points) Let $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$ be linearly *dependent* vectors in $\mathbb{R}^n$ for some $n \geq 3$. Assume furthermore that $\vec{x}_1$ and $\vec{x}_2$ are linearly independent, but that $\vec{x}_3 \in \operatorname{Span}\{\vec{x}_1, \vec{x}_2\}$.

Normally, one would not apply the Gram-Schmidt process to $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$, since they are not a basis for a subspace of $\mathbb{R}^n$.

However, what would happen if one did apply the Gram-Schmidt process to $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$? Will the process fail? If so, how? Explain.

It may (or may not) be helpful to recall the formulas used in Gram-Schmidt:

$$\begin{aligned} \vec{v}_1 &= \vec{x}_1 \\ \vec{v}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 \\ \vec{v}_3 &= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 \end{aligned}$$

Let $W = \operatorname{Span}\{\vec{x}_1, \vec{x}_2\}$. Then also $W = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$, and so

$$\vec{v}_3 = \vec{x}_3 - \operatorname{proj}_W \vec{x}_3 = \vec{0} ,$$

because we are already given that $\vec{x}_3$ is in W.

4. (15 points) Determine (for the Method of Undetermined Coefficients) the form of a particular solution to the differential equation

$$y'' + 4y = 8t\cos 2t + \sin 2t + \cos t + te^t \ .$$

(Do not solve for the coefficients.)

The characteristic equation of the associated homogeneous equation is $r^2 + 4$, which has roots $\pm 2i$. Therefore the terms involving $\cos 2t$ and $\sin 2t$ need to be multiplied by t. So, the form of the trial solution is

$$t(c_1 t + c_2)\cos 2t + t(c_3 t + c_4)\sin 2t + c_5\cos t + c_6 \sin t + (c_7 t + c_8)e^t \ .$$

5. (25 points) (a). Find a general solution to

$$y'''' + 2y'' + y = 0 \ .$$

The characteristic polynomial is $r^4 + 2r^2 + 1 = (r^2+1)^2$. This has roots $\pm i$; each is a double root.

Therefore a general solution is

$$c_1\cos t + c_2\sin t + c_3 t\cos t + c_4 t\sin t \ .$$

(b). Express the differential equation $y'''' + 2y'' + y = 0$ in matrix notation.

Let $x_1 = y$, $x_2 = y'$, $x_3 = y''$, and $x_4 = y'''$. Then:

$$\begin{aligned} x_1' &= y' = x_2 \ ; \\ x_2' &= y'' = x_3 \ ; \\ x_3' &= y''' = x_4 \ ; \\ x_4' &= y'''' = -2y'' - y = -2x_3 - x_1 \ . \end{aligned}$$

Therefore the equation in matrix form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} .$$

(c). Choose *one* solution from your answer in part (a), and express it as a solution to the system of linear equations you found in part (b).

If we choose $y = \cos t$, then $\vec{x} = \begin{bmatrix} y \\ y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \\ -\cos t \\ \sin t \end{bmatrix}$.